

TAUTOLOGICAL INTEGRALS ON CURVILINEAR HILBERT SCHEMES

GERGELY BÉRCZI, OXFORD

ABSTRACT. We take a new look at the curvilinear Hilbert scheme of points on a smooth projective variety X as a projective completion of the non-reductive quotient of holomorphic map germs from the complex line into X by polynomial reparametrisations. Using an algebraic model of this quotient coming from global singularity theory we develop an iterated residue formula for tautological integrals over curvilinear Hilbert schemes.

1. INTRODUCTION

Let X be a smooth projective variety of dimension n and let F be a rank r algebraic vector bundle on X . Let $X^{[k]}$ denote the Hilbert scheme of length k subschemes of X and let $F^{[k]}$ be the corresponding tautological rank rk bundle on $X^{[k]}$ whose fibre at $\xi \in X^{[k]}$ is $H^0(\xi, F|_\xi)$.

Let $\text{Hilb}_0^k(\mathbb{C}^n)$ be the punctual Hilbert scheme defined as the closed subset of $(\mathbb{C}^n)^{[k]} = \text{Hilb}^k(\mathbb{C}^n)$ parametrising subschemes supported at the origin. Following Rennemo [34] we define punctual geometric subsets as constructible subsets $Q \subseteq \text{Hilb}_0^k(\mathbb{C}^n)$ which are union of isomorphism classes of schemes, that is, if $\xi \in Q$ and $\xi' \in \text{Hilb}_0^k(\mathbb{C}^n)$ are isomorphic (they have isomorphic coordinate rings) then $\xi' \in Q$. Geometric subsets of $X^{[k]}$ of type (Q_1, \dots, Q_s) are those generated by finite unions, intersections and complements from sets of the form

$$P(Q_1, \dots, Q_s) = \{\xi \in X^{[k]} | \xi = \xi_1 \sqcup \dots \sqcup \xi_s, \xi_i \in Q_i\}.$$

For a geometric subset \mathcal{Z} let $\overline{\mathcal{Z}}$ denote its Zariski closure in $X^{[k]}$. Let $M(c_1, \dots, c_{rk})$ be a monomial in the Chern classes $c_i = c_i(F^{[k]})$ of weighted degree equal to $\dim \overline{\mathcal{Z}}$ where the weight of c_i is $2i$. If $\alpha_M \in \Omega^*(\mathcal{Z})$ is a closed compactly supported differential form representing the cohomology class of $M(c_1, \dots, c_{rk})$ then the Chern numbers

$$[\overline{\mathcal{Z}}] \cap M(c_1, \dots, c_{rk}) = \int_{\overline{\mathcal{Z}}} \alpha_M$$

are called tautological integrals of $F^{[k]}$. Rennemo [34] shows that these integrals can be expressed in terms of the Chern numbers of X and F .

Theorem 1.1 (Rennemo [34]). *Let $\mathcal{M}_{r,n}$ denote the set of weighted-degree- n monomials in the Chern classes $c_1(F), \dots, c_r(F)$ and $c_1(X), \dots, c_n(X)$. For $S \in \mathcal{M}_{r,n}$ let $\alpha_S \in \Omega^{top}(X)$ be a closed compactly supported differential form representing the cohomology class of S and let $y_S = \int_X \alpha_S$ denote the corresponding intersection number. Let $\mathcal{Z} \subset X^{[k]}$ be a geometric subset. Then for any Chern monomial $M = M(c_1, \dots, c_{rk})$ of weighted degree $\dim \overline{\mathcal{Z}}$ there is a polynomial R_M in $|\mathcal{M}_{r,n}|$ variables depending only on M such that*

$$[\overline{\mathcal{Z}}] \cap M(c_1, \dots, c_{rk}) = R_M(y_S : S \in \mathcal{M}_{r,n}).$$

The proof of [34] is nonconstructive and based on the fact that an element in the cohomology ring of a Grassmannian is a polynomial in the Chern classes of the universal bundle. Lacking a method of obtaining information about this polynomial, there is no apparent way of turning this proof into an algorithm. Explicit expressions for tautological integrals are not known in general. On surfaces the method of [16] yields a recursion which in principle computes the universal polynomial explicitly. The top Segre classes of tautological bundles over surfaces provides an example of this problem and the conjecture of Lehn [29] has been recently proved by Marian, Oprea and Pandharipande [30] for K3 surfaces using virtual localisation.

Let X be a smooth projective variety of dimension n . This paper provides a closed iterated residue formula for tautological integrals over the simplest geometric subsets $P(Q)$ where $s = 1$ and the punctual geometric subset Q is defined as

$$Q = \{\xi \in \text{Hilb}_0^k(\mathbb{C}^n) : \mathcal{O}_\xi \simeq \mathbb{C}[z]/z^k\}.$$

We will see that \overline{Q} is an irreducible component of the punctual Hilbert scheme. Points of $P(Q)$ correspond to curvilinear subschemes on X , i.e. subschemes contained in the germ of some smooth curve on X . In other words, these are the limit points on $X^{[k]}$ where k distinct points come together along a smooth curve. We denote this curvilinear locus by $CX^{[k]}$ and its closure by $\overline{CX}^{[k]}$ which we call the curvilinear Hilbert scheme.

The main result of the present paper is the following

Theorem 1.2. *Let $k \geq 1$ and $M(x_1, \dots, x_{r(k+1)})$ be a monomial of weighted degree $\dim \overline{CX}^{[k+1]} = n + (n-1)k$ in the variables x_i of weight $2i$ for $1 \leq i \leq r(k+1)$. Let $c_i = c_i(F^{[k+1]})$ denote the i th Chern class of the tautological rank $r(k+1)$ bundle on $X^{[k+1]}$. Then*

$$[\overline{CX}^{[k+1]}] \cap M(c_1, \dots, c_{r(k+1)}) = \int_X \text{Res}_{\mathbf{z}=\infty} \frac{(-1)^{nk} \prod_{i < j} (z_i - z_j) Q_k(\mathbf{z}) M(c_i(z_i + \theta_j, \theta_j)) d\mathbf{z}}{\prod_{i+j \leq l \leq k} (z_i + z_j - z_l)(z_1 \dots z_k)^n} \prod_{i=1}^k s_X\left(\frac{1}{z_i}\right)$$

where $s_X\left(\frac{1}{z_i}\right) = 1 + \frac{s_1(X)}{z_i} + \frac{s_2(X)}{z_i^2} + \dots + \frac{s_n(X)}{z_i^n}$ is the total Segre class at $1/z_i$ and the iterated residue is equal to the coefficient of $(z_1 \dots z_k)^{-1}$ in the expansion of the rational expression in the domain $z_1 \ll \dots \ll z_k$. Finally $Q_k(\mathbf{z})$ is a homogeneous polynomial invariant of Morin singularities given as the equivariant Poincaré dual of a Borel orbit defined below under the explanation.

Explanation and features of the residue formula:

- The iterated residue gives a degree n symmetric polynomial in Chern roots of F and Segre classes of X reproving Theorem 1.1 This shows that the dependence on Chern classes of X in fact can be expressed via the Segre classes of X .
- For fixed k the formula gives a universal generating series for the integrals as the dimension increases.
- The Chern class $c_i(z_i + \theta_j, \theta_j)$ is the coefficient of t^i in

$$c(F^{[k+1]})(t) = \prod_{j=1}^r (1 + \theta_j t) \prod_{i=1}^k \prod_{j=1}^r (1 + z_i t + \theta_j t),$$

that is, the i th Chern class of the bundle with formal Chern roots $\theta_j, z_i + \theta_j$.

- The quick description of Q_k is the following. The GL_k -module of 3-tensors $\mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2 \mathbb{C}^k)$ has a diagonal decomposition

$$\mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2 \mathbb{C}^k) = \bigoplus \mathbb{C} q_l^{mr}, \quad 1 \leq m, r, l \leq k,$$

where the T_k -weight of q_l^{mr} is $(z_m + z_r - z_l)$. Let

$$\epsilon = \sum_{m=1}^k \sum_{r=1}^{k-m} q_{m+r}^{mr} \subset W = \bigoplus_{1 \leq m+r \leq l \leq k} \mathbb{C} q_l^{mr} \subset \mathrm{Hom}(\mathbb{C}^k, \mathrm{Sym}^2 \mathbb{C}^k).$$

Then $Q_k(\mathbf{z}) = \mathrm{eP}[\overline{B_k \epsilon}, W]$ is the equivariant Poincaré dual of the Borel orbit $\overline{B_k \epsilon}$ in W , see §7.1 for details. The list of these polynomials begins as follows:

$$Q_1 = Q_2 = Q_3 = 1, \quad Q_4 = 2z_1 + z_2 - z_4.$$

In principle, Q_k may be calculated for each concrete k using a computer algebra program, but at the moment, we do not have an efficient algorithm for performing such calculations for large k and Q_k is known for $k \leq 6$, see §7.1.

The intersection theory of the Hilbert scheme of points on surfaces has been extensively studied and it can be approached from different directions. One is the inductive recursions set up in [16], an other possibility is using Nakajima calculus [33, 29]. By these methods, the integration of tautological classes is reduced to a combinatorial problem. Another strategy is to prove an equivariant version of Lehn's conjecture for the Hilbert scheme of points of \mathbb{C}^2 via appropriately weighted sums over partitions. More recently Marian, Oprea and Pandharipande proved a conjecture of Lehn [29] on integrals of top Segre classes of tautological bundles over the Hilbert schemes of points over surfaces in the K3 case via virtual localisation on the Quot schemes of the surface.

In this paper we suggest a new approach by taking a look at Hilbert schemes of points from a different perspective. We work in arbitrary dimension and not just over surfaces. Of course, for $n \geq 3$ not much is known about the irreducible components and singularities of the punctual Hilbert scheme $\mathrm{Hilb}_0^k(\mathbb{C}^n)$ so we only focus on the curvilinear component. The crucial observation is that for $k \geq 1$ the punctual curvilinear locus $CX_p^{[k+1]}$ at $p \in X$ can be described as the non-reductive quotient of k -jets of holomorphic map germs $(\mathbb{C}, 0) \rightarrow (X, p)$ by polynomial reparametrisations of \mathbb{C} at the origin. If u, v are positive integers let $J_k(u, v)$ denote the vector space of k -jets of holomorphic maps $(\mathbb{C}^u, 0) \rightarrow (\mathbb{C}^v, 0)$ at the origin, that is, the set of equivalence classes of maps $f : (\mathbb{C}^u, 0) \rightarrow (\mathbb{C}^v, 0)$ with $f' \neq 0$, where $f \sim g$ if and only if $f^{(j)}(0) = g^{(j)}(0)$ for all $j = 1, \dots, k$. One can compose map-jets via substitution and elimination of terms of degree greater than k ; this leads to the composition maps

$$(1) \quad J_k(u, v) \times J_k(v, w) \rightarrow J_k(u, w).$$

In particular, if $J_k^{\mathrm{reg}}(u, v)$ denotes the jets $f = (f', \dots, f^{(k)})$ with $f' \neq 0$ (the regular jets) then (1) defines an action of the reparametrisation group $J_k^{\mathrm{reg}}(1, 1)$ on the regular jets $J_k^{\mathrm{reg}}(1, n)$. The punctual curvilinear locus (as a set) can be identified with the quasi-projective quotient

$$CX_p^{[k+1]} \simeq J_k^{\mathrm{reg}}(1, n) / J_k^{\mathrm{reg}}(1, 1)$$

and the curvilinear Hilbert scheme is a fibrewise projective compactification of this non-reductive quotient over X as p moves on X .

Using an algebraic model coming from global singularity theory (we call this the test curve model) we reinterpret the natural embedding of the punctual Hilbert scheme $X_p^{[k+1]} = \text{Hilb}_0^{k+1}(\mathbb{C}^n)$ into the Grassmannian of codimension k subspaces in the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$ as a parametrised map

$$\phi^{\text{Grass}} : \overline{CX}_p^{[k+1]} \hookrightarrow \text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n) \text{ where } \text{Sym}^{\leq k} \mathbb{C}^n = \bigoplus_{i=1}^k \text{Sym}^i \mathbb{C}^n.$$

We then apply a two-step equivariant localisation on the fibre $\overline{CX}_p^{[k+1]}$ following the strategy of [9]. However, for tautological integrals we need to modify the proof in [9] in two crucial points:

- First, the main obstacle to apply localisation directly is that we don't know which fixed points of the ambient Grassmannian sit in the image $\overline{CX}_p^{[k+1]}$. However, for $k+1 \leq n$ we prove in [9] a residue vanishing theorem which tells that after transforming the localisation formula into an iterated residue only one distinguished fixed point of the torus action contributes to the sum. This mysterious property remains valid for tautological integrals but its proof needs a more detailed study of the rational differential form.
- Second, we need to extend the formula to the domain where $k+1 > n$, that is, the number of points is larger than the dimension. The trick here is to increase the dimension of the variety and study $\text{Hilb}_0^{k+1}(\mathbb{C}^n)$ as a subvariety of $\text{Hilb}_0^{k+1}(\mathbb{C}^{k+1})$.

The developed method reflects a surprising feature of curvilinear Hilbert schemes: in order to evaluate tautological integrals and make the residue vanishing principle work we need to increase the dimension of the variety first and work in the range where the number of points does not exceed the dimension.

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2. TAUTOLOGICAL INTEGRALS

Let X be a smooth projective variety of dimension n and let F be a rank r bundle (loc. free sheaf) on X . Let

$$X^{[k]} = \{\xi \subset X : \dim(\xi) = 0, \text{length}(\xi) = \dim H^0(\xi, \mathcal{O}_\xi) = k\}$$

denote the Hilbert scheme of k points on X parametrizing length k subschemes of X and $F^{[k]}$ the corresponding rank rk bundle on $X^{[k]}$ whose fibre over $\xi \in X^{[k]}$ is $F \otimes \mathcal{O}_\xi = H^0(\xi, F|_\xi)$.

Equivalently, $F^{[k]} = q_* p^*(F)$ where p, q denote the projections from the universal family of subschemes \mathcal{U} to X and $X^{[k]}$ respectively:

$$\begin{array}{ccc} X^{[k]} \times X \supset \mathcal{U} & \xrightarrow{q} & X^{[k]} \\ \downarrow p & & \\ X & & \end{array}$$

For simplicity let $\text{Hilb}_0^k(\mathbb{C}^n)$ denote the punctual Hilbert scheme of k points on \mathbb{C}^n defined as the closed subset of $\text{Hilb}^k(\mathbb{C}^n)$ parametrising subschemes supported at the origin. Following Rennemo [34] we define punctual geometric subsets to be the constructible subsets of the punctual Hilbert scheme containing all 0-dimensional schemes of given isomorphism types.

Definition 2.1. *A punctual geometric set is a constructible subset $Q \subseteq \text{Hilb}_0^k(\mathbb{C}^n)$ which is the union of isomorphism classes of subschemes, that is, if $\xi \in Q$ and $\xi' \in \text{Hilb}_0^k(\mathbb{C}^n)$ are isomorphic schemes then $\xi' \in Q$.*

Definition 2.2. *For an s -tuple (Q_1, \dots, Q_s) of punctual geometric sets such that $Q_i \subseteq \text{Hilb}_0^{k_i}(\mathbb{C}^n)$ and $k = \sum k_i$ define*

$$P(Q_1, \dots, Q_s) = \{\xi \in X^{[k]} : \xi = \xi_1 \sqcup \dots \sqcup \xi_s \text{ where } \xi_i \in X_{p_i}^{[k_i]} \cap Q_i \text{ for distinct } p_1, \dots, p_s\} \subseteq X^{[k]}.$$

A subset $\mathcal{Z} \subseteq X^{[k]}$ is geometric if it can be expressed as finite union, intersection and complement of sets of the form $P(Q_1, \dots, Q_s)$.

A straightforward way to produce punctual geometric subsets is by taking a complex algebra A of complex dimension k and define the corresponding

$$Q_A = \{\xi \in X^{[k]} : \mathcal{O}_\xi \simeq A\}.$$

When $A = \mathbb{C}[z]/z^k$ then $Q_A = CX_p^{[k]}$ is the punctual curvilinear locus defined in the next section and

$$\overline{CX}^{[k]} = \cup_{p \in X} \overline{CX}_p^{[k]}$$

is the curvilinear Hilbert scheme, the central object of this paper.

In this paper we work with singular homology and cohomology with rational coefficients. For a smooth manifold X the degree of a class $\eta \in H_*(X)$ means its pushforward to $H_*(pt) = \mathbb{Q}$. By choosing $\alpha_\eta \in \Omega^{top}(X)$, a closed compactly supported differential form representing the cohomology class η this degree is equal to the integral

$$\mu \cap [X] = \int_X \mu.$$

Let $\mathcal{Z} \subset X^{[k]}$ be a geometric subset with closure $\overline{\mathcal{Z}}$ and $M(c_1, \dots, c_{rk})$ be a monomial in the Chern classes $c_i = c_i(F^{[k]})$ of weighted degree equal to $\dim \overline{\mathcal{Z}}$ where the weight of c_i is $2i$. By choosing $\alpha_M \in \Omega^*(X^{[k]})$, a closed compactly supported differential form representing the cohomology class of $M(c_1, \dots, c_{rk})$, the degree

$$[\overline{\mathcal{Z}}] \cap M(c_1, \dots, c_{rk}) = \int_{\overline{\mathcal{Z}}} \alpha_M$$

is called a tautological integral of $F^{[k]}$. Here the integral of α_M on the smooth part of \overline{Z} is absolutely convergent and by definition we denote this by $\int_{\overline{Z}} \alpha_M$.

Remark 2.3. Recall (see e.g. [12]) that if $f : X \rightarrow Y$ is a smooth proper map between connected oriented manifolds such that f restricted to some open subset of X is a diffeomorphism, then for a compactly supported form μ on Y , we have $\int_X f^* \mu = \int_Y \mu$. The analogous statement for singular varieties is the following. Let $f : M \rightarrow N$ be a smooth proper map between smooth quasiprojective varieties and assume that $X \subset M$ and $Y \subset N$ are possibly singular closed subvarieties, such that f restricted to X is a birational map from X to Y . If μ is a closed differential form on N then the integral of μ on the smooth part of Y is absolutely convergent; we denote this by $\int_Y \mu$. With this convention we again have $\int_X f^* \mu = \int_Y \mu$.

In particular this means that the integral $\int_Y \mu$ of the compactly supported form μ on N is the same as the integral $\int_{\tilde{Y}} f^* \mu$ of the pull-back form $f^* \mu$ over any (partial) resolution $f : (\tilde{Y}, \tilde{M}) \rightarrow (Y, M)$.

In §4 we construct an embedding $\overline{CX}^{[k+1]}_p \subset \text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$ into a smooth Grassmannian and for $k \leq n$ we construct a partial resolution $\widehat{CX}^{[k]}_p \rightarrow \overline{CX}^{[k]}_p$. In §5 we develop the iterated residue formula of Theorem 1.2 using equivariant localisation to compute $\int_{\overline{CX}^{[k]}} P(c_i(F^{[n]}))$ which is according to the remark above equal to $\int_{\overline{CX}^{[k]}} P(c_i(F^{[n]}))$.

3. CURVILINEAR HILBERT SCHEMES

In this section we describe a geometric model for curvilinear Hilbert schemes. Let X be a smooth projective variety of dimension n and let

$$X^{[k]} = \{\xi \subset X : \dim(\xi) = 0, \text{length}(\xi) = \dim H^0(\xi, \mathcal{O}_\xi) = k\}$$

denote the Hilbert scheme of k points on X parametrizing all length k subschemes of X . For $p \in X$ let

$$X_p^{[k]} = \{\xi \in X^{[k]} : \text{supp}(\xi) = p\}$$

denote the punctual Hilbert scheme consisting of subschemes supported at p . If $\rho : X^{[k]} \rightarrow S^k X$, $\xi \mapsto \sum_{p \in X} \text{length}(\mathcal{O}_{\xi,p}) p$ denotes the Hilbert-Chow morphism then $X_p^{[k]} = \rho^{-1}(kp)$.

Definition 3.1. A subscheme $\xi \in X_p^{[k]}$ is called *curvilinear* if ξ is contained in some smooth curve $C \subset X$. Equivalently, one might say that \mathcal{O}_ξ is isomorphic to the \mathbb{C} -algebra $\mathbb{C}[z]/z^k$. The punctual curvilinear locus at $p \in X$ is the set of curvilinear subschemes supported at p :

$$CX_p^{[k]} = \{\xi \in X_p^{[k]} : \xi \subset C_p \text{ for some smooth curve } C \subset X\} = \{\xi \in X_p^{[k]} : \mathcal{O}_\xi \simeq \mathbb{C}[z]/z^k\}.$$

For surfaces ($n = 2$) $CX_p^{[k]}$ is an irreducible quasi-projective variety of dimension $n - 1$ which is an open dense subset in $X_p^{[k]}$ and therefore its closure is the full punctual Hilbert scheme at p , that is, $\overline{CX}_p^{[k]} = X_p^{[k]}$. When $n \geq 3$ the punctual Hilbert scheme $X_p^{[k]}$ is not necessarily irreducible or reduced, but the closure of the curvilinear locus is one of its irreducible components:

Lemma 3.2. $\overline{CX_p^{[k]}}$ is an irreducible component of the punctual Hilbert scheme $X_p^{[k]}$ of dimension $(n-1)(k-1)$.

Proof. Note that $\xi \in \text{Hilb}_0^{[k]}(\mathbb{C}^n)$ is not curvilinear if and only if \mathcal{O}_ξ does not contain elements of degree k , that is, after fixing some local coordinates x_1, \dots, x_n of \mathbb{C}^n at the origin we have

$$\mathcal{O}_\xi \simeq \mathbb{C}[x_1, \dots, x_n]/I \text{ for some } I \supseteq (x_1, \dots, x_n)^k.$$

This is a closed condition and therefore curvilinear subschemes can't be approximated by non-curvilinear subschemes in $\text{Hilb}_0^{[k]}(\mathbb{C}^n)$. The dimension of $\overline{CX_p^{[k]}}$ will come from the description of it as a non-reductive quotient in the next subsection. \square

Note that any curvilinear subscheme contains only one subscheme for any given smaller length and any small deformation of a curvilinear subscheme is again locally curvilinear.

Remark 3.3. Fix coordinates x_1, \dots, x_n on \mathbb{C}^n . Recall that the defining ideal I_ξ of any subscheme $\xi \in \text{Hilb}_0^{k+1}(\mathbb{C}^n)$ is a codimension k subspace in the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. The dual of this is a k -dimensional subspace S_ξ in $\mathfrak{m}^* \simeq \text{Sym}^{\leq k} \mathbb{C}^n$ giving us a natural embedding $\varphi : X_p^{[k+1]} \hookrightarrow \text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$. In what follows, we give an explicit parametrization of this embedding using an algebraic model coming from global singularity theory.

3.1. Test curve model for $CX_p^{[k]}$. If u, v are positive integers let $J_k(u, v)$ denote the vector space of k -jets of holomorphic maps $(\mathbb{C}^u, 0) \rightarrow (\mathbb{C}^v, 0)$ at the origin, that is, the set of equivalence classes of maps $f : (\mathbb{C}^u, 0) \rightarrow (\mathbb{C}^v, 0)$, where $f \sim g$ if and only if $f^{(j)}(0) = g^{(j)}(0)$ for all $j = 1, \dots, k$.

If we fix local coordinates z_1, \dots, z_u at $0 \in \mathbb{C}^u$ we can again identify the k -jet of f with the set of derivatives at the origin, that is $(f'(0), f''(0), \dots, f^{(k)}(0))$, where $f^{(j)}(0) \in \text{Hom}(\text{Sym}^j \mathbb{C}^u, \mathbb{C}^v)$. This way we get the equality

$$(2) \quad J_k(u, v) = \bigoplus_{j=1}^k \text{Hom}(\text{Sym}^j \mathbb{C}^u, \mathbb{C}^v).$$

One can compose map-jets via substitution and elimination of terms of degree greater than k ; this leads to the composition maps

$$(3) \quad J_k(u, v) \times J_k(v, w) \rightarrow J_k(u, w), \quad (\Psi_1, \Psi_2) \mapsto \Psi_2 \circ \Psi_1 \text{ modulo terms of degree } > k.$$

When $k = 1$, $J_1(u, v)$ may be identified with u -by- v matrices, and (3) reduces to multiplication of matrices.

The k -jet of a curve $(\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$ is simply an element of $J_k(1, n)$. We call such a curve γ *regular*, if $\gamma'(0) \neq 0$; introduce the notation $J_k^{\text{reg}}(1, n)$ for the set of regular curves:

$$J_k^{\text{reg}}(1, n) = \{\gamma \in J_k(1, n); \gamma'(0) \neq 0\}$$

Let $\xi \in CX_p^{[k+1]}$ be a curvilinear subscheme. It is contained in a unique smooth curve germ C_p

$$\xi \subset C_p \subset X.$$

Let $f_\xi : (\mathbb{C}, 0) \rightarrow (X, p)$ be a k -jet of germ parametrising C_p . Then $f_\xi \in J_k^{\text{reg}}(1, n)$ is determined only up to polynomial reparametrisation germs $\phi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ and therefore we get

Lemma 3.4. *The curvilinear locus $CX_p^{[k+1]}$ is equal (as a set) to the set of k -jet of regular germs at the origin of \mathbb{C}^n modulo reparametrisation:*

$$CX_p^{[k+1]} = \{k\text{-jets } (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)\} / \{k\text{-jets } (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)\} = J_k^{\text{reg}}(1, n) / J_k^{\text{reg}}(1, 1).$$

We can explicitly write out this reparametrisation action as follows; let $f_\xi(z) = zf'(0) + \frac{z^2}{2!}f''(0) + \dots + \frac{z^k}{k!}f^{(k)}(0) \in J_k^{\text{reg}}(1, n)$ a k -jet of germ at the origin (i.e no constant term) in \mathbb{C}^n with $f^{(i)} \in \mathbb{C}^n$ such that $f' \neq 0$ and let $\varphi(z) = \alpha_1 z + \alpha_2 z^2 + \dots + \alpha_k z^k \in J_k^{\text{reg}}(1, 1)$ with $\alpha_i \in \mathbb{C}, \alpha_1 \neq 0$. Then

$$(4) \quad \begin{aligned} f \circ \varphi(z) &= (f'(0)\alpha_1)z + (f'(0)\alpha_2 + \frac{f''(0)}{2!}\alpha_1^2)z^2 + \dots + \left(\sum_{i_1+\dots+i_l=k} \frac{f^{(l)}(0)}{l!} \alpha_{i_1} \dots \alpha_{i_l} \right) z^k = \\ &= (f'(0), \dots, f^{(k)}(0)/k!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \dots & 2\alpha_1\alpha_{k-1} + \dots \\ 0 & 0 & \alpha_1^3 & \dots & 3\alpha_1^2\alpha_{k-2} + \dots \\ 0 & 0 & 0 & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \alpha_1^k \end{pmatrix} \end{aligned}$$

where the (i, j) entry is $p_{i,j}(\bar{\alpha}) = \sum_{a_1+a_2+\dots+a_i=j} \alpha_{a_1}\alpha_{a_2}\dots\alpha_{a_i}$.

Remark 3.5. *The linearisation of the action of $J_k^{\text{reg}}(1, 1)$ on $J_k^{\text{reg}}(1, n)$ given as the matrix multiplication in (4) represents $J_k^{\text{reg}}(1, 1)$ as a upper triangular matrix group in $\text{GL}(n)$. It is parametrised along its first row with the free parameters $\alpha_1, \dots, \alpha_k$ and the other entries are certain (weighted homogeneous) polynomials in these free parameters. It is a \mathbb{C}^* extension of its maximal unipotent radical*

$$J_k^{\text{reg}}(1, 1) = U \rtimes \mathbb{C}^*$$

where U is the subgroup we get via substituting $\alpha_1 = 1$ and the diagonal \mathbb{C}^* acts with weights $0, 1, \dots, n-1$ on the Lie algebra $\text{Lie}(U)$. In [7] and [8] we study actions of groups of this type in a more general context.

Fix an integer $N \geq 1$ and define

$$\Theta_k = \left\{ \Psi \in J_k(n, N) : \exists \gamma \in J_k^{\text{reg}}(1, n) : \Psi \circ \gamma = 0 \right\},$$

that is, Θ_k is the set of those k -jets of germs on \mathbb{C}^n at the origin which vanish on some regular curve. By definition, Θ_k is the image of the closed subvariety of $J_k(n, N) \times J_k^{\text{reg}}(1, n)$ defined by the algebraic equations $\Psi \circ \gamma = 0$, under the projection to the first factor. If $\Psi \circ \gamma = 0$, we call γ a *test curve* of Θ .

Remark 3.6. *The subset Θ_k is the closure of an important singularity class in the jet space $J_k(n, N)$. These are called Morin singularities and the equivariant dual of Θ_k in $J_k(n, N)$ is called the Thom polynomial of Morin singularities, see [9] for details.*

Test curves of germs are generally not unique. A basic but crucial observation is the following. If γ is a test curve of $\Psi \in \Theta_k$, and $\varphi \in J_k^{\text{reg}}(1, 1)$ is a holomorphic reparametrisation of \mathbb{C} , then $\gamma \circ \varphi$ is, again, a test curve of Ψ :

$$\mathbb{C} \xrightarrow{\varphi} \mathbb{C} \xrightarrow{\gamma} \mathbb{C}^n \xrightarrow{\Psi} \mathbb{C}^N$$

$$\Psi \circ \gamma = 0 \Rightarrow \Psi \circ (\gamma \circ \varphi) = 0$$

In fact, we get all test curves of Ψ in this way if the following open dense property holds: the linear part of Ψ has 1-dimensional kernel. Before stating this in Theorem 3.8 below, let us write down the equation $\Psi \circ \gamma = 0$ in coordinates in an illustrative case. Let $\gamma = (\gamma', \gamma'', \dots, \gamma^{(k)}) \in J_k^{\text{reg}}(1, n)$ and $\Psi = (\Psi', \Psi'', \dots, \Psi^{(k)}) \in J_k(n, N)$ be the k -jets of the test curve γ and the map Ψ respectively. Using the chain rule and the notation $v_i = \gamma^{(i)}/i!$, the equation $\Psi \circ \gamma = 0$ reads as follows for $k = 4$:

$$(5) \quad \begin{aligned} \Psi'(v_1) &= 0, \\ \Psi'(v_2) + \Psi''(v_1, v_1) &= 0, \\ \Psi'(v_3) + 2\Psi''(v_1, v_2) + \Psi'''(v_1, v_1, v_1) &= 0, \\ \Psi'(v_4) + 2\Psi''(v_1, v_3) + \Psi''(v_2, v_2) + 3\Psi'''(v_1, v_1, v_2) + \Psi''''(v_1, v_1, v_1, v_1) &= 0. \end{aligned}$$

Lemma 3.7 ([22, 9]). *Let $\gamma = (\gamma', \gamma'', \dots, \gamma^{(k)}) \in J_k^{\text{reg}}(1, n)$ and $\Psi = (\Psi', \Psi'', \dots, \Psi^{(k)}) \in J_k(n, N)$ be k -jets. Then substituting $v_i = \gamma^{(i)}/i!$, the equation $\Psi \circ \gamma$ is equivalent to the following system of k linear equations with values in \mathbb{C}^N :*

$$(6) \quad \sum_{\tau \in \mathcal{P}(m)} \Psi(\mathbf{v}_\tau) = 0, \quad m = 1, 2, \dots, k.$$

Here $\mathcal{P}(m)$ denotes the set of partitions $\tau = 1^{\tau_1} \dots m^{\tau_m}$ of m into nonnegative integers and $\mathbf{v}_\tau = v_1^{\tau_1} \dots v_m^{\tau_m}$.

For a given $\gamma \in J_k^{\text{reg}}(1, n)$ and $1 \leq i \leq k$ let $\mathcal{S}_\gamma^{i,N}$ denote the set of solutions of the first i equations in (6), that is,

$$(7) \quad \mathcal{S}_\gamma^{i,N} = \{\Psi \in J_k(n, N), \Psi \circ \gamma = 0 \text{ up to order } i\}$$

The equations (6) are linear in Ψ , hence

$$\mathcal{S}_\gamma^{i,N} \subset J_k(n, N)$$

is a linear subspace of codimension iN , i.e a point of $\text{Grass}_{\text{codim}=iN}(J_k(n, N))$, whose dual, $(\mathcal{S}_\gamma^{i,N})^*$, is an iN -dimensional subspace of $J_k(n, N)^*$. These subspaces are invariant under the reparametrization of γ . In fact, $\Psi \circ \gamma$ has N vanishing coordinates and therefore

$$\mathcal{S}_\gamma^{i,N} = \mathcal{S}_\gamma^{i,1} \otimes \mathbb{C}^N$$

holds.

For $\Psi \in J_k(n, N)$ let $\Psi^1 \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^N)$ denote the linear part. When $N \geq n$ then the subset

$$\tilde{\mathcal{S}}_\gamma^{i,N} = \{\Psi \in \mathcal{S}_\gamma^{i,N} : \dim \ker \Psi^1 = 1\}$$

is an open dense subset of the subspace $\mathcal{S}_\gamma^{i,N}$. In fact it is not hard to see that the complement $\tilde{\mathcal{S}}_\gamma^{i,N} \setminus \mathcal{S}_\gamma^{i,N}$ where the kernel of Ψ^1 has dimension at least two is a closed subvariety of codimension $N - n + 2$.

Note that for $N = 1$, according to (2), the dual space $J_k(n, 1)^*$ can be and will be identified with

$$\text{Hom}(\mathbb{C}, \text{Sym}^{\leq n}(\mathbb{C}^n)^*) \simeq \text{Sym}^{\leq k} \mathbb{C}^n$$

where $\text{Sym}^{\leq k} \mathbb{C}^n = \bigoplus_{i=1}^k \text{Sym}^i \mathbb{C}^n$ and we identified \mathbb{C}^n with its dual.

Theorem 3.8. (1) *The map*

$$\phi : J_k^{\text{reg}}(1, n) \rightarrow \text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$$

defined as $\gamma \mapsto (S_\gamma^{k,1})^$ is $J_k^{\text{reg}}(1, 1)$ -invariant and induces an injective map on the $J_k^{\text{reg}}(1, 1)$ -orbits into the Grassmannian*

$$\phi^{\text{Grass}} : CX_p^{[k+1]} = J_k^{\text{reg}}(1, n)/J_k^{\text{reg}}(1, 1) \hookrightarrow \text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n).$$

Moreover, ϕ and ϕ^{Grass} are $\text{GL}(n)$ -equivariant with respect to the standard action of $\text{GL}(n)$ on $J_k^{\text{reg}}(1, n) \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ and the induced action on $\text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$.

(2) *The image of ϕ and the image of φ defined in Remark 3.5 coincide in $\text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$:*

$$\text{im}(\phi) = \text{im}(\varphi) \subset \text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n).$$

Proof. For the first part it is enough to prove that for $\Psi \in \Theta_k$ with $\dim \ker \Psi^1 = 1$ and $\gamma, \delta \in J_k^{\text{reg}}(1, n)$

$$\Psi \circ \gamma_1 = \Psi \circ \gamma_2 = 0 \Leftrightarrow \exists \Delta \in J_k^{\text{reg}}(1, 1) \text{ such that } \gamma = \delta \circ \Delta.$$

We prove this statement by induction. Let $\gamma = v_1 t + \dots + v_k t^k$ and $\delta = w_1 t + \dots + w_k t^k$. Since $\dim \ker \Psi^1 = 1$, $v_1 = \lambda w_1$, for some $\lambda \neq 0$. This proves the $k = 1$ case.

Suppose the statement is true for $k - 1$. Then, using the appropriate order- $(k - 1)$ diffeomorphism, we can assume that $v_m = w_m$, $m = 1 \dots k - 1$. It is clear then from the explicit form (6) (cf. (5)) of the equation $\Psi \circ \gamma = 0$, that $\Psi^1(v_k) = \Psi^1(w_k)$, and hence $w_k = v_k - \lambda v_1$ for some $\lambda \in \mathbb{C}$. Then $\gamma = \Delta \circ \delta$ for $\Delta = t + \lambda t^k$, and the proof is complete.

The second part immediately follows from the definition of φ and ϕ . \square

Remark 3.9. (1) *In particular the second part of Theorem 3.8 tells us that the curvilinear component*

$$\overline{CX_p^{[k+1]}} = \overline{\text{im}(\phi)} \subset \text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$$

has a $\text{GL}(n)$ -equivariant embedding into the Grassmannian $\text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$ as the closure of the image of ϕ .

(2) *For a point $\gamma \in J_k^{\text{reg}}(1, n)$ let $v_i = \frac{\gamma^{(i)}}{i!} \in \mathbb{C}^n$ denote the normed i th derivative. Then from Lemma 3.7 immediately follows that for $1 \leq i \leq k$ (see [9]):*

$$(8) \quad S_\gamma^{i,1} = \text{Span}_{\mathbb{C}}(v_1, v_2 + v_1^2, \dots, \sum_{\tau \in \mathcal{P}(i)} v_\tau) \subset \text{Sym}^{\leq k} \mathbb{C}^n.$$

This explicit parametrisation of the curvilinear component is crucial in building our localisation process in the next section.

(3) *Since ϕ is $\text{GL}(n)$ -equivariant, for $k \leq n$ the $\text{GL}(n)$ -orbit of*

$$p_{n,k} = \phi(e_1, \dots, e_k) = \text{Span}_{\mathbb{C}}(e_1, e_2 + e_1^2, \dots, \sum_{\tau \in \mathcal{P}(k)} e_\tau),$$

forms a dense subset of the image $J_k^{\text{reg}}(1, n)$ and therefore

$$\overline{CX_p^{[k+1]}} = \overline{\phi(J_k^{\text{reg}}(1, n))} = \overline{\text{GL}_n \cdot p_{n,k}}.$$

3.2. Jet bundles and $CX^{[k]}$. Let X be a smooth projective variety and let $J_k X \rightarrow X$ denote the bundle of k -jets of germs of parametrized curves in X ; its fibre over $x \in X$ is the set of equivalence classes of germs of holomorphic maps $f : (\mathbb{C}, 0) \rightarrow (X, x)$, with the equivalence relation $f \sim g$ if and only if the derivatives $f^{(j)}(0) = g^{(j)}(0)$ are equal for $0 \leq j \leq k$. If we choose local holomorphic coordinates (z_1, \dots, z_n) on an open neighbourhood $\Omega \subset X$ around x , the elements of the fibre $J_k X_x$ are represented by the Taylor expansions

$$f(t) = x + tf'(0) + \frac{t^2}{2!}f''(0) + \dots + \frac{t^k}{k!}f^{(k)}(0) + O(t^{k+1})$$

up to order k at $t = 0$ of \mathbb{C}^n -valued maps $f = (f_1, f_2, \dots, f_n)$ on open neighbourhoods of 0 in \mathbb{C} . Locally in these coordinates the fibre can be written as

$$J_k X_x = \{(f'(0), \dots, f^{(k)}(0)/k!)\} = (\mathbb{C}^n)^k,$$

which we identify with $J_k(1, n)$. Note that $J_k X$ is not a vector bundle over X since the transition functions are polynomial but not linear, see [13] for details.

Let $J_k^{\text{reg}} X$ denote the bundle of k -jets of germs of parametrized regular curves in X , that is, where the first derivative $f' \neq 0$ is nonzero. Its fibre is isomorphic with $J_k^{\text{reg}}(1, n)$.

$J_k^{\text{reg}}(1, 1)$ acts fibrewise on the jet bundle $J_k^{\text{reg}} X$ and the full curvilinear component $CX^{[k]}$ on X can be identified with the non-reductive fibrewise quotient of $J_k^{\text{reg}} X$ by $J_k^{\text{reg}}(1, 1)$:

$$CX^{[k+1]} = J_k^{\text{reg}} X / J_k^{\text{reg}}(1, 1).$$

More precisely, introduce the notation

$$\text{Sym}^{\leq k} T_X^* = T_X^* \oplus \text{Sym}^2(T_X^*) \oplus \dots \oplus \text{Sym}^k(T_X^*)$$

for the vector bundle on X whose fibre is isomorphic to $\text{Sym}^{\leq k} \mathbb{C}^n$. The Grassmannian bundle $\text{Grass}_k(\text{Sym}^{\leq k} T_X^*)$ and the jet bundle $J_k^{\text{reg}} X$ have an induced fibrewise action of $\text{GL}(n)$ and we have the following fibrewise version of Theorem 4.1

Corollary 3.10. *The quotient $J_k^{\text{reg}} X / J_k^{\text{reg}}(1, 1)$ has the structure of a locally trivial bundle over X , and Theorem 4.1 gives us a $\text{GL}(n)$ -equivariant holomorphic embedding*

$$\phi^{\text{Grass}} : CX^{[k+1]} = J_k^{\text{reg}} X / J_k^{\text{reg}}(1, 1) \hookrightarrow \text{Grass}_k(\text{Sym}^{\leq k} T_X^*)$$

into the Grassmannian bundle of $\text{Sym}^{\leq k} T_X^$ over X . The fibrewise compactification*

$$\overline{CX}^{[k+1]} = \overline{\phi^{\text{Grass}}(J_k^{\text{reg}} X)}$$

of the image is the curvilinear component of the Hilbert scheme of $k+1$ points on X .

3.3. Tautological bundles over $\overline{CX}^{[k]}$. Let F be a rank r vector bundle over X . The fibre of the corresponding rank $r(k+1)$ tautological bundle $F^{[k+1]}$ on $\overline{CX}^{[k+1]}$ at the point ξ is

$$F_{\xi}^{[k+1]} = H^0(\xi, F|_{\xi}) = H^0(\mathcal{O}_{\xi} \otimes F).$$

Using our embedding $\phi^{\text{Grass}} : \overline{CX}^{[k+1]} \hookrightarrow \text{Grass}_k(\text{Sym}^{\leq k} T_X^*)$ this fibre can be identified as

$$F_{\xi}^{[k+1]} = (\mathcal{O}_{\text{Grass}_k(\text{Sym}^{\leq k} T_X^*)} \oplus \mathcal{E})_{\phi(\xi)} \otimes F_{\text{supp}(\xi)}$$

where \mathcal{E} is the tautological rank k bundle over $\text{Grass}_k(\text{Sym}^{\leq k} T_X^*)$. Hence the total Chern class of $F^{[k+1]}$ can be written as

$$c(F^{[k+1]}) = \prod_{j=1}^r (1 + \theta_j) \prod_{i=1}^k \prod_{j=1}^r (1 + \eta_i + \theta_j)$$

where $c(F) = \prod_{j=1}^r (1 + \theta_j)$ and $c(\mathcal{E}) = \prod_{i=1}^k (1 + \eta_i)$ are the Chern classes for the corresponding bundles. In particular the Chern class

$$(9) \quad c_i(F^{[k+1]}) = C_i(c_1(\mathcal{E}), \dots, c_k(\mathcal{E}), c_1(F), \dots, c_r(F))$$

can be expressed as a polynomial function C_i in Chern classes of \mathcal{E} and F .

4. PARTIAL RESOLUTIONS OF $\overline{CX}^{[k+1]}$

In this section first we construct a partial resolution of the (highly singular) punctual curvilinear component $\overline{CX}_p^{[k+1]} \subset \text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$ in two steps. The first partial resolution is defined for any choice of parameters n, k and it uses nested Hilbert schemes. For the second step we need to impose the very restrictive condition $k \leq n$, that is the number of points can't exceed the dimension of the variety plus 1. We will see how to dispose this condition in Section §7.

4.1. Completion in nested Hilbert schemes. Let

$$X^{[k_1, \dots, k_t]} = \{(\xi_1 \subset \xi_2 \subset \dots \subset \xi_t) : \xi_i \in X^{[k_i]}\} \subset X^{[k_1]} \times \dots \times X^{[k_t]}$$

be the nested Hilbert scheme defining flags of subschemes of length vector (k_1, \dots, k_t) .

Curvilinear subschemes contain only one subscheme for any given smaller length. Therefore $\xi \in CX_p^{[k+1]}$ defines a unique flag

$$\mathcal{F}(\xi) = (\xi_1 \subset \xi_2 \subset \dots \subset \xi_k) \in CX_p^{[2]} \times \dots \times CX_p^{[k+1]} \subset X^{[2, \dots, k+1]}$$

where ξ_i is defined via

$$\mathcal{O}_{\xi_i} = \mathcal{O}_{\xi} / \mathcal{O}_{X,p}^{i+1} \simeq \mathbb{C}[z] / z^{i+1}$$

and therefore $\xi_i \in CX_p^{[i+1]}$. This defines an embedding

$$\tilde{\phi} : CX_p^{[k+1]} \hookrightarrow X^{[2, \dots, k+1]}$$

$$\xi \mapsto (\xi_1 \subset \dots \subset \xi_k).$$

Let $f_{\xi} \in J_k^{\text{reg}}(1, n)$ denote the k -jet corresponding to $\xi \in CX_p^{[k+1]}$ and let $\mathcal{S}_{\xi}^i = \mathcal{S}_{f_{\xi}}^{i,1} \subset J_k(n, 1)$ be the solution space defined in (7) where $N = 1$. Then $\tilde{\phi}$ can be equivalently written as

$$f_{\xi} \mapsto ((\mathcal{S}_{\xi}^1)^* \subset (\mathcal{S}_{\xi}^2)^* \subset \dots \subset (\mathcal{S}_{\xi}^k)^*) \in \text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$$

or using coordinates as

$$f = f_{\xi} \mapsto \text{Span}_{\mathbb{C}}(f') \subset \text{Span}_{\mathbb{C}}(f', f'' + (f')^2) \subset \dots \subset \text{Span}_{\mathbb{C}}(f', f'' + (f')^2, \dots, f^{[k]} + \sum_{\Sigma a_i = k} (f^{[i]})^{a_i}).$$

Theorem 3.8 has the following immediate

Corollary 4.1. *The map*

$$\tilde{\phi} : J_k^{\text{reg}}(1, n) \rightarrow \text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$$

$$\gamma \mapsto \mathcal{F}_\gamma = ((S_\gamma^1)^* \subset \dots \subset (S_\gamma^k)^*)$$

is $J_k^{\text{reg}}(1, 1)$ -invariant and induces an injective map on the $J_k^{\text{reg}}(1, 1)$ -orbits into the flag manifold

$$\phi^{\text{Flag}} : CX_p^{[k+1]} = J_k^{\text{reg}}(1, n)/J_k^{\text{reg}}(1, 1) \hookrightarrow \text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^n).$$

Moreover, all these maps are $\text{GL}(n)$ -equivariant with respect to the standard action of $\text{GL}(n)$ on $J_k^{\text{reg}}(1, n) \subset \text{Hom}(\mathbb{C}^k, \mathbb{C}^n)$ and the induced action on $\text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$.

Let $\widehat{CX}_p^{[k+1]}$ denote the closure of $\tilde{\phi}(J_k^{\text{reg}}(1, n))$ in $\text{Flag}(k, \text{Sym}^{\leq k} \mathbb{C}^n)$.

4.2. Blowing up along the linear part. Assume $k \leq n$. Let $\pi : J_k(n, 1)^* \simeq \text{Sym}^{\leq k} \mathbb{C}^n = \oplus_{i=1}^k \text{Sym}^i \mathbb{C}^n \rightarrow \mathbb{C}^n$ denote the projection to the first (linear) factor and define

$$\widehat{CX}_p^{[k+1]} = \{((S_\gamma^1)^* \subset \dots \subset (S_\gamma^k)^*), (V_1 \subset \dots \subset V_k) : \pi(S_\gamma^i)^* \subset V_i\} \subset \widehat{CX}_p^{[k+1]} \times \text{Flag}_k(\mathbb{C}^n).$$

Equivalently, let $P_{k,n} \subset \text{GL}_n$ denote the parabolic subgroup which preserves the flag

$$\mathbf{f} = (\text{Span}(e_1) \subset \text{Span}(e_1, e_2) \subset \dots \subset \text{Span}(e_1, \dots, e_k) \subset \mathbb{C}^n).$$

and $\mathfrak{p}_{k,n} = \tilde{\phi}(e_1, \dots, e_k)$ the base point in $\text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$. Define the partial resolution $\widehat{CX}_p^{[k+1]}$ of $\widehat{CX}_p^{[k+1]}$ as the fibrewise compactification of $\widehat{CX}_p^{[k+1]}$ on $\text{Flag}_k(\mathbb{C}^n) = \text{GL}(n)/P_{k,n}$, that is,

$$\widehat{CX}_p^{[k+1]} = \text{GL}(n) \times_{P_{k,n}} \overline{P_{k,n} \cdot \mathfrak{p}_{k,n}} \rightarrow \overline{\text{GL}(n) \cdot \mathfrak{p}_{k,n}} = \widehat{CX}_p^{[k+1]}$$

with the resolution map $\widehat{CX}_p^{[k+1]} \rightarrow \widehat{CX}_p^{[k+1]}$ given by $\rho(g, z) = g \cdot z$.

The geometric resolutions $\widehat{CX}_p^{[k+1]}$ and $\widehat{CX}_p^{[k+1]}$ of $CX_p^{[k+1]}$ constructed in this section form the fibres of partial resolution bundles $\widehat{CX}^{[k]}$ and $\widehat{CX}^{[k]}$ over X with partial resolution maps

$$\widehat{CX}^{[k]} \rightarrow \widehat{CX}^{[k]} \rightarrow \overline{CX}^{[k]}$$

where

$$\widehat{CX}^{[k]} = \overline{\tilde{\phi}(J_k(T_X^*))} \subset \text{Flag}_k(\text{Sym}^{\leq k} T_X^*)$$

is the closure of the fibrewise embedding $CX^{[k]} = J_k^{\text{reg}} X / J_k^{\text{reg}}(1, 1) \hookrightarrow \text{Flag}_k(\text{Sym}^{\leq k} T_X^*)$ and for $k \leq n$

$$\widehat{CX}^{[k+1]} = \{((S_1 \subset \dots \subset S_k), (V_1 \subset \dots \subset V_k) : \pi(S_i) \subset V_i\} \subset \widehat{CX}^{[k+1]} \times \text{Flag}_k(T_X^*).$$

Here $\pi : J_k^{\text{reg}} X \simeq \oplus_{i=1}^k \text{Sym}^i T_X^* \rightarrow T_X^*$ denotes again the projection to the first factor. The fibre of $\widehat{CX}^{[k+1]}$ over $p \in X$ is $\widehat{CX}_p^{[k+1]}$ and therefore $\widehat{CX}^{[k+1]}$ canonically sits in $\text{Flag}_k(\text{Sym}^{\leq k} T_X^*)$.

5. EQUIVARIANT LOCALISATION ON $\widetilde{CX}^{[k+1]}$

Let F be a rank r vector bundle over X and let $F^{[k+1]}$ denote the corresponding rank $(k+1)r$ tautological bundle over $X^{[k+1]}$. We use the same notation $F^{[k+1]}$ for its pull-back along the partial resolution map $\widetilde{CX}^{[k+1]} \rightarrow \overline{CX}^{[k+1]}$. Fix a Chern polynomial $P(c_1, \dots, c_{r(k+1)})$ of weighted degree $\dim \overline{CX}^{[k+1]} = n + (n-1)k$ where $c_i = c_i(F^{[k+1]})$ are the Chern classes of the tautological bundle. In this section we start developing an iterated residue formula for the tautological integral $\int_{\overline{CX}^{[k]}} P$. This formula is attained via a two-step equivariant localisation process and it is crucially based on a vanishing theorem of residues.

5.1. Equivariant de-Rham model and the Atiyah-Bott formula. This section is a short introduction to equivariant cohomology and localisation. For more details, we refer the reader to Section 2 of [9] and [23].

Let G be a compact Lie group with Lie algebra \mathfrak{g} and let M be a C^∞ manifold endowed with the action of G . The G -equivariant differential forms are defined as differential form valued polynomial functions on \mathfrak{g} :

$$\Omega_G^\bullet(M) = \{\alpha : \mathfrak{g} \rightarrow \Omega^\bullet(M) : \alpha(gX) = g\alpha(X) \text{ for } g \in G, X \in \mathfrak{g}\} = (S^\bullet \mathfrak{g}^* \otimes \Omega^\bullet(M))^G$$

where $(g \cdot \alpha)(X) = g \cdot (\alpha(g^{-1} \cdot X))$. One can define equivariant the exterior differential d_G on $(S^\bullet \mathfrak{g}^* \otimes \Omega^\bullet(M))^G$ by the formula

$$(d_G \alpha)(X) = (d - \iota(X_M))\alpha(X),$$

where $\iota(X_M)$ denotes the contraction by the vector field X_M . This increases the degree of an equivariant form by one if the \mathbb{Z} -grading is given on $(S^\bullet \mathfrak{g}^* \otimes \Omega^\bullet(M))^G$ by $\deg(P \otimes \alpha) = 2 \deg(P) + \deg(\alpha)$ for $P \in S^\bullet \mathfrak{g}^*, \alpha \in \Omega^\bullet(M)$. The homotopy formula $\iota(X)d + d\iota(X) = \mathcal{L}(X)$ implies that $d_G^2(\alpha)(X) = -\mathcal{L}(X)\alpha(X) = 0$ for any $\alpha \in (S^\bullet \mathfrak{g}^* \otimes \Omega^\bullet(M))^G$, and therefore $(d_G, \Omega_G^\bullet(M))$ is a complex. The equivariant cohomology $H_G^*(M)$ of the G -manifold M is the cohomology of the complex $(d_G, \Omega_G^\bullet(M))$. Note that $\alpha \in \Omega_G^\bullet(M)$ is equivariantly closed if and only if

$$\alpha(X) = \alpha(X)^{[0]} + \dots + \alpha(X)^{[n]} \text{ such that } \iota(X_M)\alpha(X)^{[i]} = d\alpha(X)^{[i-2]}.$$

Here $\alpha(X)^{[i]} \in \Omega^i(M)$ is the degree- i part of $\alpha(X) \in \Omega^\bullet(M)$ and $\alpha^{[i]} : \mathfrak{g} \rightarrow \Omega^i(M)$ is a polynomial function.

The equivariant push-forward map $\int_M : \Omega_G(M) \rightarrow (S^\bullet \mathfrak{g}^*)^G$ is defined by the formula

$$(10) \quad \left(\int_M \alpha \right)(X) = \int_M \alpha(X) = \int_M \alpha^{[n]}(X)$$

When the n -dimensional complex torus $T = (\mathbb{C}^*)^n$ acts on M let $K = U(1)^n$ be its maximal unipotent subgroup and $\mathfrak{k} = \text{Lie}(K)$ its Lie algebra. We define the T -equivariant cohomology $H_T^\bullet(M)$ to be the $H_K^\bullet(M)$, the equivariant DeRham cohomology defined by the action of K . If $M_0(X)$ is the zero locus of the vector field X_M , then the form $\alpha(X)^{[n]}$ is exact outside $M_0(X)$. (see Proposition 7.10 in [23]), and this suggests that the integral $\int_M \alpha(X)$ depends only on the restriction $\alpha(X)|_{M_0(X)}$.

Theorem 5.1 (Atiyah-Bott [2], Berline-Vergne [11]). *Suppose that M is a compact manifold and T is a complex torus acting smoothly on M , and the fixed point set M^T of the T -action on M is finite. Then for any cohomology class $\alpha \in H_T^\bullet(M)$*

$$\int_M \alpha = \sum_{f \in M^T} \frac{\alpha^{[0]}(f)}{\text{Euler}^T(T_f M)}.$$

Here $\text{Euler}^T(T_f M)$ is the T -equivariant Euler class of the tangent space $T_f M$, and $\alpha^{[0]}$ is the differential-form-degree-0 part of α .

The right hand side in the localisation formula considered in the fraction field of the polynomial ring of $H_T^\bullet(\text{point}) = H^\bullet(BT) = S^\bullet \mathfrak{t}^*$ (see more on details in [2, 10]). Part of the statement is that the denominators cancel when the sum is simplified.

5.2. Equivariant Poincaré duals and multidegrees. The Atiyah-Bott formula works for holomorphic actions of tori on nonsingular projective varieties. In our case, however, the punctual curvilinear component $\overline{CX}_p^{[k+1]}$ is highly singular at the fixed points so the AB localisation does not apply directly as the equivariant Euler class of the tangent space at a singular fixed point is not well defined. But $\overline{CX}_p^{[k+1]}$ sits in the nonsingular ambient space $\text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$ and an intuitive idea would be to put $\text{Euler}^T(T_f \text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n))$ into the denominator on the right hand side which we then compensate in the numerator with some sort of dual of the tangent cone of $\overline{CX}_p^{[k+1]}$ at f sitting in the tangent space of $\text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$ at f . This idea indeed works and it becomes incarnate in the Rossman formula in §5.3.

Let $T = (\mathbb{C}^*)^n$ be a complex torus with $K = U(1)^n$ its maximal compact subgroup and $\mathfrak{t} = \text{Lie}(K)$ its Lie algebra. Let M be a manifold endowed with a T action. The compactly supported equivariant cohomology groups $H_{K,\text{cpt}}^\bullet(M)$ are obtained by restricting the equivariant de Rham complex to compactly supported (or quickly decreasing at infinity) differential forms. Clearly $H_{K,\text{cpt}}^\bullet(M)$ is a module over $H_K^\bullet(M)$. When $M = W$ is an N -dimensional complex vector space, and the action is linear, one has $H_K^\bullet(W) = S^\bullet \mathfrak{t}^*$ and $H_{K,\text{cpt}}^\bullet(W)$ is a free module over $H_K^\bullet(W)$ generated by a single element of degree $2N$:

$$(11) \quad H_{K,\text{cpt}}^\bullet(W) = H_K^\bullet(W) \cdot \text{Thom}_K(W),$$

called the Thom class of W .

A T -invariant algebraic subvariety Σ of dimension d in W represents a T -equivariant $2d$ -cycle in the sense that

- a compactly-supported equivariant form μ of degree $2d$ is absolutely integrable over the components of maximal dimension of Σ , and $\int_\Sigma \mu \in S^\bullet \mathfrak{t}$;
- if $d_K \mu = 0$, then $\int_\Sigma \mu$ depends only on the class of μ in $H_{K,\text{cpt}}^\bullet(W)$,
- and $\int_\Sigma \mu = 0$ if $\mu = d_K \nu$ for a compactly-supported equivariant form ν .

Definition 5.2. *Let Σ be an T -invariant algebraic subvariety of dimension d in the vector space W . Then the equivariant Poincaré dual of Σ is the polynomial on \mathfrak{t} defined by the integral*

$$(12) \quad \text{eP}[\Sigma, W] = \frac{1}{(2\pi)^d} \int_\Sigma \text{Thom}_K(W).$$

An immediate consequence of the definition is that for an equivariantly closed differential form μ with compact support, we have

$$\int_{\Sigma} \mu = \int_W \mathrm{eP}[\Sigma, W] \cdot \mu.$$

This formula serves as the motivation for the term *equivariant Poincaré dual*. This definition naturally extends to the case of an analytic subvariety of \mathbb{C}^n defined in the neighborhood of the origin, or more generally, to any T -invariant cycle in \mathbb{C}^n .

Note that $\mathrm{eP}[\Sigma, W]$ is determined by the maximal dimensional components of Σ and in fact it can be characterised and axiomatised by some of its basic properties. These are carefully stated in [9] Proposition 2.3 and proofs can be found in [36],[39],[31], the list reads as: positivity, additivity on maximal dimensional component, deformation invariance, symmetry and finally a formula for complete intersections of hypersurfaces. These properties provide an algorithm for computing $\mathrm{eP}[\Sigma, W]$ as follows (see [31] §8.5 and [9, 6] for details): we pick any monomial order on the coordinates of W and apply Groebner deformation on the ideal of Σ to deform it onto its initial monomial ideal. The spectrum of this monomial ideal is the union of some coordinate subspaces in W with multiplicities whose equivariant dual is then given as the sum of the duals of the maximal dimensional subspaces by the additivity property. For these linear subspaces the formula for complete intersections has the following special form. Let $W = \mathrm{Spec}(\mathbb{C}[y_1, \dots, y_N])$ acted on by the n -dimensional torus T diagonally where the weight of y_i is η_i . Then for every subset $\mathbf{i} \subset \{1, \dots, N\}$ we have

$$(13) \quad \mathrm{eP}[\{y_i = 0, i \in \mathbf{i}\}, W] = \prod_{i \in \mathbf{i}} \eta_i.$$

The weights η_1, \dots, η_N are linear forms of the basis elements $\lambda_1, \dots, \lambda_n$ of \mathfrak{t}^* . Let $\mathrm{coeff}(\eta_i, j)$ denote the coefficient of λ_j in η_i ($1 \leq i \leq N, 1 \leq j \leq n$) and introduce the notation

$$\deg(\eta_1, \dots, \eta_N; m) = \#\{i; \mathrm{coeff}(\eta_i, m) \neq 0\}.$$

Let $\Sigma \subset W$ be a T -invariant subvariety. It is clear from the formula (13) that the λ_m -degree of $\mathrm{eP}[\Sigma, W]$ satisfies

$$(14) \quad \deg_{\lambda_m} \mathrm{eP}[\Sigma, W] \leq \deg(\eta_1, \dots, \eta_N; m)$$

for any $1 \leq m \leq n$.

Finally we state one of the basic properties listed in [9] Proposition 2.3 as a lemma here as this will be used repeatedly later.

Lemma 5.3 (Elimination property, [9] Prop 2.3). *Let $\Sigma \subset W$ be a closed T -invariant subvariety and denote by $I(\Sigma)$ the ideal of functions vanishing on Σ . Fix a polynomial $f \in \mathbb{C}[y_1, \dots, y_N]$ of weight η_0 , and let Σ_f be the variety in $W \oplus \mathbb{C}y_0$ with ideal generated by $I(\Sigma)$ and $y_0 - f$. Then*

$$\mathrm{eP}[\Sigma_f, W \oplus \mathbb{C}y_0] = \eta_0 \cdot \mathrm{eP}[\Sigma, W]$$

Example 5.4. *Let $W = \mathbb{C}^4$ endowed with a $T = (\mathbb{C}^*)^3$ -action, whose weights η_1, η_2, η_3 and η_4 span \mathfrak{t}^* , and satisfy $\eta_1 + \eta_3 = \eta_2 + \eta_4$. Choose $p = (1, 1, 1, 1) \in W$; then the affine toric variety*

$$\overline{T \cdot p} = \{(y_1, y_2, y_3, y_4) \in \mathbb{C}^4; y_1 y_3 = y_2 y_4\}.$$

is a hypersurface and its equivariant dual is given by the weight of the equation:

$$\mathrm{eP}[\overline{T \cdot p}, W] = \eta_1 + \eta_3 = \eta_2 + \eta_4.$$

An other way to see this is to fix the monomial order $>$ induced from $y_1 > y_2 > y_3 > y_4$, then the ideal $I = (y_1 y_3 - y_2 y_4)$ has initial ideal $\mathrm{in}_I = (y_1 y_3)$ whose spectrum is the union of the hyperplanes $\{y_1 = 0\}$ and $\{y_3 = 0\}$ with duals η_1, η_3 respectively.

Remark 5.5. An alternative and slightly more general topological definition of the equivariant dual is the following, see [21, 27, 15] for details. For a Lie group G let $EG \rightarrow BG$ be a right principal G -bundle with EG contractible. Such a bundle is universal in the topological setting: if $E \rightarrow B$ is any principal G -bundle, then there is a map $B \rightarrow BG$, unique up to homotopy, such that E is isomorphic to the pullback of EG . If X is a smooth algebraic G -variety then the topological definition of the G -equivariant cohomology of X is

$$H_G^*(X) = H^*(EG \times_G X).$$

If Y is a G -invariant subvariety then Y represents a G -equivariant cohomology class in the equivariant cohomology of X , namely the ordinary Poincaré dual of $EG \times_G Y$ in $EG \times_G X$. This is the equivariant dual of Y in X :

$$\mathrm{eP}[Y, X] = \mathrm{PD}(EG \times_G Y, EG \times_G X).$$

5.3. The Rossman formula. Let Z be a complex manifold with a holomorphic T -action, and let $M \subset Z$ be a T -invariant analytic subvariety with an isolated fixed point $p \in M^T$. Then one can find local analytic coordinates near p , in which the action is linear and diagonal. Using these coordinates, one can identify a neighborhood of the origin in $T_p Z$ with a neighborhood of p in Z . We denote by $\hat{T}_p M$ the part of $T_p Z$ which corresponds to M under this identification; informally, we will call $\hat{T}_p M$ the T -invariant *tangent cone* of M at p . This tangent cone is not quite canonical: it depends on the choice of coordinates; the equivariant dual of $\Sigma = \hat{T}_p M$ in $W = T_p Z$, however, does not. Rossman named this the *equivariant multiplicity* of M in Z at p :

$$(15) \quad \mathrm{emult}_p[M, Z] \stackrel{\mathrm{def}}{=} \mathrm{eP}[\hat{T}_p M, T_p Z].$$

Remark 5.6. In the algebraic framework one might need to pass to the tangent scheme of M at p (cf. [20]). This is canonically defined, but we will not use this notion.

The analog of the Atiyah-Bott formula for singular subvarieties of smooth ambient manifolds is the following

Proposition 5.7 (Rossman's localisation formula [36]). *Let $\mu \in H_T^*(Z)$ be an equivariant class represented by a holomorphic equivariant map $\mathfrak{t} \rightarrow \Omega^\bullet(Z)$. Then*

$$(16) \quad \int_M \mu = \sum_{p \in M^T} \frac{\mathrm{emult}_p[M, Z]}{\mathrm{Euler}^T(T_p Z)} \cdot \mu^{[0]}(p),$$

where $\mu^{[0]}(p)$ is the differential-form-degree-zero component of μ evaluated at p .

5.4. Equivariant localisation on $\widetilde{CX}^{[k+1]}$ for $k \leq n$. In this subsection we start to develop a two step equivariant localisation method on $\widetilde{CX}^{[k+1]}$ using the Rossmann formula. As the partial resolution $\widetilde{CX}^{[k+1]}$ described in §4.2 is defined only for $k \leq n$ we impose this condition in this section.

Recall from §4.2 the blow-up definition

$$\widetilde{CX}_p^{[k+1]} = \mathrm{GL}(n) \times_{P_{k,n}} \overline{P_{k,n} \cdot \mathfrak{p}_{k,n}} \rightarrow \overline{CX}_p^{[k+1]}$$

sitting in $\mathrm{Flag}_k(\mathrm{Sym}^{\leq k} \mathbb{C}^n)$ which fibres over the flag manifold $\mathrm{GL}(n)/P_{k,n} = \mathrm{Flag}_k(\mathbb{C}^n)$:

$$\begin{array}{ccc} \widetilde{CX}_p^{[k+1]} & \xrightarrow{\rho} & \overline{CX}_p^{[k+1]} \subset \mathrm{Flag}_k(\mathrm{Sym}^{\leq k} \mathbb{C}^n) \\ \downarrow \mu & & \\ \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^n)/B_k & = & \mathrm{Flag}_k(\mathbb{C}^n) \end{array}$$

and the fibres of μ are isomorphic to $\overline{P_{k,n} \cdot p_k} \subset \mathrm{Flag}_k(\mathrm{Sym}^{\leq k} \mathbb{C}^n)$. The corresponding fibred version of this diagram over X gives the partial resolution of the curvilinear Hilbert scheme $\widetilde{CX}^{[k+1]} \rightarrow \overline{CX}^{[k+1]}$:

$$(17) \quad \begin{array}{ccc} \widetilde{CX}^{[k+1]} & \xrightarrow{\rho} & \overline{CX}^{[k+1]} \subset \mathrm{Flag}_k(\mathrm{Sym}^{\leq k} T_X^*) \\ \downarrow \mu & & \\ \mathrm{Flag}_k(T_X^*) & & \\ \downarrow \tau & & \\ X & & \end{array}$$

where $\mathrm{Flag}_k(T_X^*)$ is the flag bundle of the cotangent bundle T_X^* , and over every point $p \in X$ we get back the previous diagram, that is, the fibres of $\tilde{\pi} = \tau \circ \mu : \widetilde{CX}^{[k+1]} \rightarrow X$ are canonically isomorphic to $\widetilde{CX}_p^{[k+1]}$.

Fix a Chern polynomial $P = P(c_1, \dots, c_{r(k+1)})$ of degree $\dim \overline{CX}^{[k+1]} = n + (n-1)k$ where $c_i = c_i(F^{[k+1]})$ are the Chern classes of the tautological rank $r(k+1)$ bundle on the curvilinear Hilbert scheme. To evaluate the integral $\int_{\overline{CX}^{[k+1]}} P$ we can first integrate (push forward) along the fibres of $\tilde{\pi} : \widetilde{CX}^{[k+1]} \rightarrow X$ followed by integration over X . These fibres are canonically isomorphic to $\widetilde{CX}_p^{[k+1]}$ endowed with a natural $\mathrm{GL}(n)$ action induced by the standard $\mathrm{GL}(n)$ action on \mathbb{C}^n and we can use this action to perform torus equivariant localisation on $\widetilde{CX}_p^{[k+1]}$ to integrate along the fibres. Recall that $K = U(1)^n$ is the maximal compact subgroup of the maximal complex torus T of $\mathrm{GL}(n, \mathbb{C})$ and $\mathfrak{t} = \mathrm{Lie}(K)$. Take a fibrewise equivariant extension

$$\alpha \in H_T^f = S^\bullet \mathfrak{t}^* \otimes (\Omega^\bullet(\widetilde{CX}_p^{[k+1]})^K \otimes \Omega^\bullet(X))$$

with respect to the torus action on $\widetilde{CX}_p^{[k+1]}$. Then α is a polynomial function on \mathfrak{t} with values in the $\Omega^\bullet(X)$ -module $\Omega^\bullet(\widetilde{CX}_p^{[k+1]})^K \otimes \Omega^\bullet(X)$. Integration along the fibre is the map

$$H_T^f \rightarrow S^\bullet \mathfrak{t}^* \otimes \Omega^\bullet(X)$$

defined as

$$\left(\int \alpha \right)(X) = \int_{\widetilde{CX}_p^{[k+1]}} \alpha^{[\dim(\widetilde{CX}_p^{[k+1]})]}(X) \text{ for all } X \in \mathfrak{t}$$

where $\alpha^{[\dim(\widetilde{CX}_p^{[k+1]})]}$ is the $(\Omega^\bullet(\widetilde{CX}_p^{[k+1]})^K)$ -degree- d part of α with $d = \dim(\widetilde{CX}_p^{[k+1]})$. In short, we consider the $\Omega^\bullet(X)$ part of α as a constant and apply the standard localisation map (10) on $S^\bullet \mathfrak{t}^* \otimes (\Omega^\bullet(\widetilde{CX}_p^{[k+1]})^K)$.

Note that $\mu : \widetilde{CX}_p^{[k+1]} \rightarrow \text{Flag}_k(\mathbb{C}^n)$ gives a $\text{GL}(n)$ -equivariant fibration over the flag manifold $\text{Flag}_k(\mathbb{C}^n)$. Let $e_1, \dots, e_n \in \mathbb{C}^n$ be an eigenbasis of \mathbb{C}^n for the T action on $\widetilde{CX}_p^{[k+1]}$ with weights $\lambda_1, \dots, \lambda_n \in \mathfrak{t}^*$ and let

$$\mathbf{f} = (\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1, \dots, e_k \rangle \subset \mathbb{C}^n)$$

denote the standard flag in \mathbb{C}^n fixed by the parabolic $P_{k,n} \subset \text{GL}(n)$. Since the torus action on $\widetilde{CX}_p^{[k+1]}$ is obtained by the restriction of a $\text{GL}(n)$ -action to its subgroup of diagonal matrices T_n , the Weyl group of permutation matrices S_n acts transitively on the fixed points set $\text{Flag}_k(\mathbb{C}^n)^{T_n}$ taking the standard flag \mathbf{f} to $\sigma(\mathbf{f})$ and Proposition 5.1 gives us

$$(18) \quad \int_{\widetilde{CX}_p^{[k+1]}} \alpha = \sum_{\sigma \in S_n / S_{n-k}} \frac{\alpha_{\sigma(\mathbf{f})}}{\prod_{1 \leq m \leq k} \prod_{i=m+1}^n (\lambda_{\sigma \cdot i} - \lambda_{\sigma \cdot m})},$$

where

- σ runs over the ordered k -element subsets of $\{1, \dots, n\}$ labeling the fixed flags $\sigma(\mathbf{f}) = (\langle e_{\sigma(1)} \rangle \subset \dots \subset \langle e_{\sigma(1)}, \dots, e_{\sigma(k)} \rangle \subset \mathbb{C}^n)$ in \mathbb{C}^n ,
- $\prod_{1 \leq m \leq k} \prod_{i=m+1}^n (\lambda_{\sigma(i)} - \lambda_{\sigma(m)})$ is the equivariant Euler class of the tangent space of $\text{Flag}_k(\mathbb{C}^n)$ at $\sigma(\mathbf{f})$,
- if $\widetilde{CX}_{\sigma(\mathbf{f})}^{[k+1]} = \mu^{-1}(\sigma(\mathbf{f}))$ denotes the fibre then $\alpha_{\sigma(\mathbf{f})} = \left(\int_{\widetilde{CX}_{\sigma(\mathbf{f})}^{[k+1]}} \alpha \right)^{[0]}(\sigma(\mathbf{f})) \in S^\bullet \mathfrak{t}^* \otimes \Omega^\bullet(X)$ is the differential-form-degree-zero part evaluated at $\sigma(\mathbf{f})$ and $\alpha_{\sigma(\mathbf{f})} = \sigma \cdot \alpha_{\mathbf{f}}$ with respect to the natural Weyl group action on $S^\bullet \mathfrak{t}^*$.

In particular, when $\alpha = \alpha(\theta_1, \dots, \theta_r, \eta_1, \dots, \eta_k)$ is a bi-symmetric polynomial in the Chern roots θ_i of the pull-back of F over $\widetilde{CX}_p^{[k+1]} \subset \text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$ and the Chern roots η_j of the tautological rank k bundle \mathcal{E} then $\alpha_{\mathbf{f}}$ is a polynomial in two sets of variables: in the basic weights $\lambda = (\lambda_1 \dots \lambda_n)$ and in the $\theta = (\theta_1 \dots \theta_r)$. Since $\mu^{-1}(\mathbf{f})$ is invariant under $P_{k,n}$ only, this polynomial is not necessarily symmetric in the λ 's. Note that $\alpha_{\mathbf{f}}$ contains only Chern roots of the tautological rank k bundle \mathcal{E} and therefore it does not depend on the last $n - k$ basic weights: $\lambda_{k+1}, \dots, \lambda_n \in \mathfrak{t}^*$.

$$\alpha_{\mathbf{f}} = \alpha_{\mathbf{f}}(\theta_1, \dots, \theta_r, \lambda_1, \dots, \lambda_k)$$

and

$$(19) \quad \alpha_{\sigma(\mathbf{f})} = \sigma \cdot \alpha_{\mathbf{f}} = \alpha_{\mathbf{f}}(\theta_1, \dots, \theta_r, \lambda_{\sigma(1)}, \dots, \lambda_{\sigma(k)}) \in S^\bullet \mathfrak{g}^* \otimes H^*(X)$$

is the σ -shift of the polynomial $\alpha_{\mathbf{f}}$ corresponding to the distinguished fixed flag \mathbf{f} .

5.5. Transforming the localisation formula into iterated residue. In this section we transform the right hand side of (18) into an iterated residue. This step turns out to be crucial in handling the combinatorial complexity of the Atiyah-Bott localisation formula and captures the symmetry of the fixed point data in an efficient way which enables us to prove the vanishing of the contribution of all but one of the fixed points.

To describe this formula, we will need the notion of an *iterated residue* (cf. e.g. [37]) at infinity. Let $\omega_1, \dots, \omega_N$ be affine linear forms on \mathbb{C}^k ; denoting the coordinates by z_1, \dots, z_k , this means that we can write $\omega_i = a_i^0 + a_i^1 z_1 + \dots + a_i^k z_k$. We will use the shorthand $h(\mathbf{z})$ for a function $h(z_1 \dots z_k)$, and $d\mathbf{z}$ for the holomorphic n -form $dz_1 \wedge \dots \wedge dz_k$. Now, let $h(\mathbf{z})$ be an entire function, and define the *iterated residue at infinity* as follows:

$$(20) \quad \text{Res}_{z_1=\infty} \text{Res}_{z_2=\infty} \dots \text{Res}_{z_k=\infty} \frac{h(\mathbf{z}) d\mathbf{z}}{\prod_{i=1}^N \omega_i} \stackrel{\text{def}}{=} \left(\frac{1}{2\pi i} \right)^k \int_{|z_1|=R_1} \dots \int_{|z_k|=R_k} \frac{h(\mathbf{z}) d\mathbf{z}}{\prod_{i=1}^N \omega_i},$$

where $1 \ll R_1 \ll \dots \ll R_k$. The torus $\{|z_m| = R_m; m = 1 \dots k\}$ is oriented in such a way that $\text{Res}_{z_1=\infty} \dots \text{Res}_{z_k=\infty} d\mathbf{z}/(z_1 \dots z_k) = (-1)^k$. We will also use the following simplified notation: $\text{Res}_{\mathbf{z}=\infty} \stackrel{\text{def}}{=} \text{Res}_{z_1=\infty} \text{Res}_{z_2=\infty} \dots \text{Res}_{z_k=\infty}$.

In practice, one way to compute the iterated residue (20) is the following algorithm: for each i , use the expansion

$$(21) \quad \frac{1}{\omega_i} = \sum_{j=0}^{\infty} (-1)^j \frac{(a_i^0 + a_i^1 z_1 + \dots + a_i^{q(i)-1} z_{q(i)-1})^j}{(a_i^{q(i)} z_{q(i)})^{j+1}},$$

where $q(i)$ is the largest value of m for which $a_i^m \neq 0$, then multiply the product of these expressions with $(-1)^k h(z_1 \dots z_k)$, and then take the coefficient of $z_1^{-1} \dots z_k^{-1}$ in the resulting Laurent series.

We repeat the proof of the following iterated residue theorem from [9].

Proposition 5.8 ([9] Proposition 5.4). *For any homogeneous polynomial $Q(\mathbf{z})$ on \mathbb{C}^k we have*

$$(22) \quad \sum_{\sigma \in \mathcal{S}_n / \mathcal{S}_{n-k}} \frac{Q(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(k)})}{\prod_{1 \leq m \leq k} \prod_{i=m+1}^n (\lambda_{\sigma(i)} - \lambda_{\sigma(m)})} = \text{Res}_{\mathbf{z}=\infty} \frac{\prod_{1 \leq m < l \leq k} (z_m - z_l) Q(\mathbf{z}) d\mathbf{z}}{\prod_{l=1}^k \prod_{i=1}^n (\lambda_i - z_l)}$$

Proof. We compute the iterated residue (22) using the Residue Theorem on the projective line $\mathbb{C} \cup \{\infty\}$. The first residue, which is taken with respect to z_k , is a contour integral, whose value is minus the sum of the z_k -residues of the form in (22). These poles are at $z_k = \lambda_j$, $j = 1 \dots n$, and after canceling the signs that arise, we obtain the following expression for the right hand side of (22):

$$\sum_{j=1}^n \frac{\prod_{1 \leq m < l \leq k-1} (z_m - z_l) \prod_{l=1}^{k-1} (z_l - \lambda_j) Q(z_1 \dots z_{k-1}, \lambda_j) dz_1 \dots dz_{k-1}}{\prod_{l=1}^{k-1} \prod_{i=1}^n (\lambda_i - z_l) \prod_{i \neq j}^n (\lambda_i - \lambda_j)}.$$

After cancellation and exchanging the sum and the residue operation, at the next step, we have

$$(-1)^{k-1} \sum_{j=1}^n \operatorname{Res}_{z_{k-1}=\infty} \frac{\prod_{1 \leq m < l \leq k-1} (z_m - z_l) Q(z_1 \dots z_{k-1}, \lambda_j) dz_1 \dots dz_{k-1}}{\prod_{i \neq j}^n ((\lambda_i - \lambda_j) \prod_{l=1}^{k-1} (\lambda_i - z_l))}.$$

Now we again apply the Residue Theorem, with the only difference that now the pole $z_{k-1} = \lambda_j$ has been eliminated. As a result, after converting the second residue to a sum, we obtain

$$(-1)^{2k-3} \sum_{j=1}^n \sum_{s=1, s \neq j}^n \frac{\prod_{1 \leq m < l \leq k-2} (z_l - z_m) Q(z_1 \dots z_{k-2}, \lambda_s, \lambda_j) dz_1 \dots dz_{k-2}}{(\lambda_s - \lambda_j) \prod_{i \neq j, s}^n ((\lambda_i - \lambda_j)(\lambda_i - \lambda_s) \prod_{l=1}^{k-1} (\lambda_i - z_l))}.$$

Iterating this process, we arrive at a sum very similar to (18). The difference between the two sums will be the sign: $(-1)^{k(k-1)/2}$, and that the $k(k-1)/2$ factors of the form $(\lambda_{\sigma(i)} - \lambda_{\sigma(m)})$ with $1 \leq m < i \leq k$ in the denominator will have opposite signs. These two differences cancel each other, and this completes the proof. \square

Remark 5.9. *Changing the order of the variables in iterated residues, usually, changes the result. In this case, however, because all the poles are normal crossing, formula (22) remains true no matter in what order we take the iterated residues.*

Proposition 5.8 together with (18) and (19) gives

Proposition 5.10. *Let $k \leq n$ and $\alpha = \alpha(\theta_1, \dots, \theta_r, \eta_1, \dots, \eta_k)$ be a bi-symmetric polynomial in the Chern roots θ_i of the pull-back of F over $\widetilde{CX}_p^{[k+1]} \subset \operatorname{Flag}_k(\operatorname{Sym}^{\leq k} \mathbb{C}^n)$ and the Chern roots η_j of the tautological rank k bundle \mathcal{E} . Then*

$$\int_{\widetilde{CX}_p^{[k+1]}} \alpha = \operatorname{Res}_{\mathbf{z}=\infty} \frac{\prod_{1 \leq m < l \leq k} (z_m - z_l) \alpha_{\mathbf{f}}(\theta_1, \dots, \theta_r, z_1, \dots, z_k) d\mathbf{z}}{\prod_{l=1}^k \prod_{i=1}^n (\lambda_i - z_l)}$$

where $s_i(\mathbf{z}) = s_i(z_1, \dots, z_k)$ denotes the i th symmetric polynomial in z_1, \dots, z_k .

Next, we proceed a second localisation on the fibre

$$\widetilde{CX}_{\mathbf{f}}^{[k+1]} = \mu^{-1}(\mathbf{f}) \simeq \overline{P_{k,n} \cdot p_k} \subset \operatorname{Flag}_k(\operatorname{Sym}^{\leq k} \mathbb{C}^n)$$

to compute $\alpha_{\mathbf{f}}(\theta, \mathbf{z})$. Since $\widetilde{CX}_{\mathbf{f}}^{[k+1]}$ is invariant under the T -action on $\operatorname{Flag}_k(\operatorname{Sym}^{\leq k} \mathbb{C}^n)$, we can apply Rossmann's integration formula, see Proposition 5.7. Note that the fibre $\widetilde{CX}_{\mathbf{f}}^{[k+1]} = \overline{P_{k,n} \cdot p_k}$ sits in the submanifold

$$\operatorname{Flag}_k^*(\operatorname{Sym}^{\leq k} \mathbb{C}^n) = \{V_1 \subset \dots \subset V_k \subset \operatorname{Sym}^{\leq k} \mathbb{C}^n : \dim(V_i) = i, V_i \subset \operatorname{Span}_{\mathbb{C}}(e_{\tau} : \Sigma \tau \leq i)\}$$

of $\operatorname{Flag}_k(\operatorname{Sym}^{\leq k} \mathbb{C}^n)$. Since the subspaces

$$W_i = \operatorname{Span}_{\mathbb{C}}(e_{\tau} : \Sigma \tau \leq i) \subset \operatorname{Sym}^{\leq k} \mathbb{C}^n$$

are invariant under the upper Borel $B_n \subset \operatorname{GL}(n)$ which fixes the flag \mathbf{f} ,

$$\operatorname{Flag}_k^*(\operatorname{Sym}^{\leq k} \mathbb{C}^n) \subset \operatorname{Flag}_k(\operatorname{Sym}^{\leq k} \mathbb{C}^n)$$

is a B_n -invariant subvariety.

We apply the Rossman formula for $M = X_{\mathbf{f}}, Z = \text{Flag}_k^*(\text{Sym}^{\leq k} \mathbb{C}^n)$ and $\mu = \alpha_{\mathbf{f}}$. The fixed points on

$$Z = \text{Flag}_k^*(\text{Sym}^{\leq k} \mathbb{C}^n) \subset \bigoplus_{i=1}^k W_1 \wedge \dots \wedge W_i$$

are parametrised by *admissible* sequences of partitions $\pi = (\pi_1, \dots, \pi_k)$. We call a sequence of partitions $\pi = (\pi_1 \dots \pi_k) \in \Pi^{\times d}$ admissible if

- (1) $\Sigma \pi_l = l$ for $1 \leq l \leq k$, and
- (2) $\pi_l \neq \pi_m$ for $1 \leq l \neq m \leq k$.

We will denote the set of admissible sequences of length k by Π_k . The corresponding fixed point is then

$$\bigoplus_{i=1}^k e_{\pi_1} \wedge \dots \wedge e_{\pi_i} \in \bigoplus_{i=1}^k W_1 \wedge \dots \wedge W_i$$

where $e_{\pi} = \prod_{j \in \pi} e \in \text{Sym}^{|\pi|} \mathbb{C}^n$.

Then the Rossman formula (16) and Proposition 5.10 give us

Proposition 5.11. *Let $k \leq n$ and let $\alpha = \alpha(\theta_1, \dots, \theta_r, \eta_1, \dots, \eta_k)$ be a bi-symmetric polynomial in the Chern roots θ_i of the pull-back of F over $\widetilde{CX}_p^{[k+1]} \subset \text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$ and the Chern roots η_j of the tautological rank k bundle \mathcal{E} . Then*

$$(23) \quad \int_{\widetilde{CX}_p^{[k+1]}} \alpha = \sum_{\pi \in \Pi_k \cap \overline{P_{k,n} \cdot p_k}} \text{Res}_{\mathbf{z}=\infty} \frac{Q_{\pi}(\mathbf{z}) \prod_{m < l} (z_m - z_l) \alpha(\theta, z_{\pi_1}, \dots, z_{\pi_k})}{\prod_{l=1}^k \prod_{\tau \leq l}^{\tau \neq \pi_1 \dots \pi_l} (z_{\tau} - z_{\pi_l}) \prod_{l=1}^k \prod_{i=1}^n (\lambda_i - z_l)} d\mathbf{z}.$$

where $Q_{\pi}(\mathbf{z}) = \text{emult}_{\pi}[X_{\mathbf{f}}, \text{Flag}_k^*]$ and $z_{\pi} = \sum_{i \in \pi} z_i$.

This formula reduces the computation of the tautological integrals $\int_{\widetilde{CX}_p^{[k+1]}} \alpha$ to determine the fixed point set $\Pi_k \cap \widetilde{CX}_{\mathbf{f}}^{[k+1]}$ and the multidegree $Q_{\pi}(\mathbf{z}) = \text{emult}_{\pi}[X_{\mathbf{f}}, \text{Flag}_k^*]$ of the tangent cone of $\widetilde{CX}_{\mathbf{f}}^{[k+1]}$ in $\text{Flag}_k^*(\text{Sym}^{\leq k} \mathbb{C}^n)$.

6. THE RESIDUE VANISHING THEOREM

The first immediate problem arising with our formula (23) is that we do not have a complete description of the fixed point set $\Pi_k \cap \widetilde{CX}_{\mathbf{f}}^{[k+1]}$ and in fact it seems to be a hard question to decide which torus fixed points on $\text{Flag}_k^*(\text{Sym}^{\leq k} \mathbb{C}^n)$ sit in the orbit closure $\widetilde{CX}_{\mathbf{f}}^{[k+1]} = \overline{P_{k,n} \cdot p_k}$. The second problem we face is how to compute the multidegrees $Q_{\pi}(\mathbf{z}) = \text{emult}_{\pi}[X_{\mathbf{f}}, \text{Flag}_k^*]$ for those admissible sequences which represent fixed points in $\overline{P_{k,n} \cdot p_k}$. We postpone this second problem to the next section and here we focus on the first question which has a particularly nice—and surprising—answer. Namely, we do not need to know which fixed points sit in $\overline{P_{k,n} \cdot p_k}$ because our limited knowledge on the equations of the $P_{k,n}$ -orbit is enough to show that all but one terms on the right hand side of (23) vanish. This key feature of the iterated residue has already appeared in [9] but here we need to prove a stronger version where the total degree of the rational forms are zero. We devote the rest of this section to the proof of

Theorem 6.1 (The Residue Vanishing Theorem). *Let $k+1 \leq n$ and $\alpha = \alpha(\theta_1, \dots, \theta_r, \eta_1, \dots, \eta_k)$ be a bi-symmetric polynomial in the Chern roots θ_i of the pull-back of F over $\widetilde{CX}_p^{[k+1]} \subset \text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$ and the Chern roots η_j of the tautological rank k bundle \mathcal{E} . Then*

(1) *All terms but the one corresponding to $\pi_{\text{dst}} = ([1], [2], \dots, [k])$ vanish in (23) leaving us with*

$$(24) \quad \int_{\widetilde{CX}_p^{[k+1]}} \alpha = \text{Res}_{\mathbf{z}=\infty} \frac{Q_{[1], \dots, [k]}(\mathbf{z}) \prod_{m < l} (z_m - z_l) \alpha(\theta, \mathbf{z})}{\prod_{\text{sum}(\tau) \leq l \leq k} (z_\tau - z_l) \prod_{l=1}^k \prod_{i=1}^n (\lambda_i - z_l)} d\mathbf{z}.$$

(2) *If $|\tau| \geq 3$ then $Q_k(\mathbf{z}) = Q_{([1], \dots, [k])}(\mathbf{z})$ is divisible by $z_\tau - z_l$ for all $l \geq \text{sum}(\tau)$ implying the simplified formula*

$$(25) \quad \int_{\widetilde{CX}_p^{[k+1]}} \alpha = \text{Res}_{\mathbf{z}=\infty} \frac{Q_k(\mathbf{z}) \prod_{m < l} (z_m - z_l) \alpha(\theta, \mathbf{z})}{\prod_{m+r \leq l \leq k} (z_m + z_r - z_l) \prod_{l=1}^k \prod_{i=1}^n (\lambda_i - z_l)} d\mathbf{z}.$$

Remark 6.2. (1) *The geometric meaning of $Q_k(\mathbf{z})$ in (25) is the following, see also [9] Theorem 6.16. Let $T_k \subset B_k \subset \text{GL}(k)$ be the subgroups of invertible diagonal and upper-triangular matrices, respectively; denote the diagonal weights of T_k by z_1, \dots, z_k . Consider the $\text{GL}(k)$ -module of 3-tensors $\text{Hom}(\mathbb{C}^k, \text{Sym}^2 \mathbb{C}^k)$; identifying the weight- $(z_m + z_r - z_l)$ symbols q_l^{mr} and q_l^{rm} , we can write a basis for this space as follows:*

$$\text{Hom}(\mathbb{C}^k, \text{Sym}^2 \mathbb{C}^k) = \bigoplus \mathbb{C} q_l^{mr}, \quad 1 \leq m, r, l \leq k.$$

Consider the point $\epsilon = \sum_{m=1}^k \sum_{r=1}^{k-m} q_{mr}^{m+r}$ in the B_k -invariant subspace

$$N_k = \bigoplus_{1 \leq m+r \leq l \leq k} \mathbb{C} q_l^{mr} \subset \text{Hom}(\mathbb{C}^k, \text{Sym}^2 \mathbb{C}^k).$$

Set the notation O_k for the orbit closure $\overline{B_k \epsilon} \subset N_k$, then $Q_k(\mathbf{z})$ is the T_k -equivariant Poincaré dual $Q_k(\mathbf{z}) = \text{eP}[O_k, N_k]_{T_k}$, which is a homogeneous polynomial of degree $\dim(N_k) - \dim(O_k)$. For small k these polynomials are the following (see [9] §7):

$$Q_2 = Q_3 = 1, Q_4 = 2z_1 + z_2 - z_4$$

$$Q_5 = (2z_1 + z_2 - z_5)(2z_1^2 + 3z_1z_2 - 2z_1z_5 + 2z_2z_3 - z_2z_4 - z_2z_5 - z_3z_4 + z_4z_5).$$

(2) *To understand the significance of this vanishing theorem we note that while the fixed point set Π_k on $\text{Flag}_k^*(\text{Sym}^{\leq k} \mathbb{C}^n)$ is well understood, it is not clear which of these fixed points sit in $X_{\mathbf{f}}$. But we have enough information to prove that none of those fixed points in $X_{\mathbf{f}}$ contribute to the iterated residue except for the distinguished fixed point $\pi_{\text{dst}} = ([1], [2], \dots, [k])$.*

(3) *The Residue Vanishing Theorem is valid under the condition $k+1 \leq n$ which is slightly stronger than the condition $k \leq n$ we worked with so far and which guaranteed the existence of $\widetilde{CX}_p^{[k+1]}$. We will remedy this condition in §7.*

Remark 6.3. *Remark 2.3 for singular varieties and ordinary compactly supported differential forms holds for compactly supported equivariant forms as follows. Let T be a complex torus and $f : M \rightarrow N$ be a smooth proper T -equivariant map between smooth quasiprojective varieties. Now assume that $X \subset M$ and $Y \subset N$ are possibly singular T -invariant closed*

subvarieties, such that f restricted to X is a birational map from X to Y . Next, let μ be an equivariantly closed differential form on N with values in polynomials on \mathfrak{t} . Then the integral of μ on the smooth part of Y is absolutely convergent; we denote this by $\int_Y \mu$. With this convention we again have

$$(26) \quad \int_X f^* \mu = \int_Y \mu,$$

and we can define integrals of equivariant forms on singular quasi-projective varieties simply by passing to any partial equivariant resolution or equivalently to integration over the smooth locus. In particular, applying this for the partial resolution $\rho : \widehat{CX}_p^{[k+1]} \rightarrow \overline{CX}_p^{[k+1]}$ we get

$$\int_{\widehat{CX}_p^{[k+1]}} \alpha = \int_{\overline{CX}_p^{[k+1]}} \rho^* \alpha$$

for any $\alpha \in \Omega^*(\overline{CX}_p^{[k+1]})$ closed compactly supported differential form.

6.1. The vanishing of residues. In this subsection following [9] §6.2 we describe the conditions under which iterated residues of the type appearing in the sum in (23) vanish and we prove Theorem 6.1.

We start with the 1-dimensional case, where the residue at infinity is defined by (20) with $d = 1$. By bounding the integral representation along a contour $|z| = R$ with R large, one can easily prove

Lemma 6.4. *Let $p(z), q(z)$ be polynomials of one variable. Then*

$$\operatorname{Res}_{z=\infty} \frac{p(z) dz}{q(z)} = 0 \quad \text{if } \deg(p(z)) + 1 < \deg(q).$$

Consider now the multidimensional situation. Let $p(\mathbf{z}), q(\mathbf{z})$ be polynomials in the k variables $z_1 \dots z_k$, and assume that $q(\mathbf{z})$ is the product of linear factors $q = \prod_{i=1}^N L_i$, as in (23). We continue to use the notation $d\mathbf{z} = dz_1 \dots dz_k$. We would like to formulate conditions under which the iterated residue

$$(27) \quad \operatorname{Res}_{z_1=\infty} \operatorname{Res}_{z_2=\infty} \dots \operatorname{Res}_{z_k=\infty} \frac{p(\mathbf{z}) d\mathbf{z}}{q(\mathbf{z})}$$

vanishes. Introduce the following notation:

- For a set of indices $S \subset \{1 \dots k\}$, denote by $\deg(p(\mathbf{z}); S)$ the degree of the one-variable polynomial $p_S(t)$ obtained from p via the substitution $z_m \rightarrow \begin{cases} t & \text{if } m \in S, \\ 1 & \text{if } m \notin S. \end{cases}$ When $p(\mathbf{z})$ is the product of linear forms, $\deg(p(\mathbf{z}); S)$ is the number of terms with nonzero coefficients in front of at least one of z_s for $s \in S$.
- For a nonzero linear form $L = a_0 + a_1 z_1 + \dots + a_k z_k$, denote by $\operatorname{coeff}(L, z_l) = a_l$ the coefficient in front of z_l ;
- finally, for $1 \leq m \leq k$, set

$$\operatorname{lead}(q(\mathbf{z}); m) = \#\{i; \max\{l; \operatorname{coeff}(L_i, z_l) \neq 0\} = m\},$$

which is the number of those factors L_i in which the coefficient of z_m does not vanish, but the coefficients of z_{m+1}, \dots, z_k are 0.

We can group the N linear factors of $q(\mathbf{z})$ according to the nonvanishing coefficient with the largest index; in particular, for $1 \leq m \leq k$ we have

$$\deg(q(\mathbf{z}); m) \geq \text{lead}(q(\mathbf{z}); m), \text{ and } \sum_{m=1}^k \text{lead}(q(\mathbf{z}); m) = N.$$

Proposition 6.5 ([9] Proposition 6.3). *Let $p(\mathbf{z})$ and $q(\mathbf{z})$ be polynomials in the variables $z_1 \dots z_k$, and assume that $q(\mathbf{z})$ is a product of linear factors: $q(\mathbf{z}) = \prod_{i=1}^N L_i$; set $d\mathbf{z} = dz_1 \dots dz_k$. Then*

$$\text{Res}_{z_1=\infty} \text{Res}_{z_2=\infty} \dots \text{Res}_{z_k=\infty} \frac{p(\mathbf{z}) d\mathbf{z}}{q(\mathbf{z})} = 0$$

if for some $l \leq k$, either of the following two options hold:

- $\deg(p(\mathbf{z}); k, k-1, \dots, l) + k - l + 1 < \deg(q(\mathbf{z}); k, k-1, \dots, l)$,
or
- $\deg(p(\mathbf{z}); l) + 1 < \deg(q(\mathbf{z}); l) = \text{lead}(q(\mathbf{z}); l)$.

Note that for the second option, the equality $\deg(q(\mathbf{z}); l) = \text{lead}(q(\mathbf{z}); l)$ means that

$$(28) \quad \text{for each } i = 1 \dots N \text{ and } m > l, \text{coeff}(L_i, z_l) \neq 0 \text{ implies } \text{coeff}(L_i, z_m) = 0.$$

We are ready to proof the Residue Vanishing Theorem. Recall that our goal is to show that all the terms of the sum in (23) vanish except for the one corresponding to $\pi_{\text{dst}} = ([1] \dots [k])$. The plan is to apply Proposition 6.5 in stages to show that the iterated residue vanishes unless $z_i = [i]$ holds, starting with $i = k$ and going backwards.

Fix a sequence $\pi = (\pi_1, \dots, \pi_k) \in \Pi_k$, and consider the iterated residue corresponding to it on the right hand side of (23). The expression under the residue is the product of two fractions:

$$\frac{p(\mathbf{z})}{q(\mathbf{z})} = \frac{p_1(\mathbf{z})}{q_1(\mathbf{z})} \cdot \frac{p_2(\mathbf{z})}{q_2(\mathbf{z})},$$

where

$$(29) \quad \frac{p_1(\mathbf{z})}{q_1(\mathbf{z})} = \frac{Q_\pi(\mathbf{z}) \prod_{m < l} (z_m - z_l)}{\prod_{l=1}^k \prod_{\substack{\tau \neq \pi_1 \dots \pi_l \\ \text{sum}(\tau) \leq l}} (z_\tau - z_{\pi_l})} \text{ and } \frac{p_2(\mathbf{z})}{q_2(\mathbf{z})} = \frac{\alpha(\theta_1, \dots, \theta_r, z_{\pi_1}, \dots, z_{\pi_k})}{\prod_{l=1}^k \prod_{i=1}^n (\lambda_i - z_l)}.$$

Note that $p(\mathbf{z})$ is a polynomial, while $q(\mathbf{z})$ is a product of linear forms. As a first step we show that if $\pi_k \neq [k]$, then already the first residue in the corresponding term on the right hand side of (23) – the one with respect to z_k – vanishes. Indeed, if $\pi_k \neq [k]$, then $\deg(q_2(\mathbf{z}); k) = n$, while z_k does not appear in $p_2(\mathbf{z})$. On the other hand, $\deg(q_1(\mathbf{z}); k) = 1$, because the only term which contains z_k is the one corresponding to $l = k, \tau = [k] \neq \pi_k$. By (14) $\deg(Q_\pi(\mathbf{z}), k) \leq 1$ holds so

$$(30) \quad \deg(p_1(\mathbf{z})p_2(\mathbf{z}); k) = k \text{ and } \deg(q_1(\mathbf{z})q_2(\mathbf{z}); k) = n + 1$$

and $k \leq n - 1$, so $\deg(p(\mathbf{z})) \leq \deg(q(\mathbf{z})) + 2$ holds and we can apply Lemma 6.4.

We can thus assume that $\pi_k = [k]$, and proceed to the next step and take the residue with respect to z_{k-1} . If $\pi_{k-1} \neq [k-1]$ then

$$(31) \quad \deg(q_2(\mathbf{z}), k-1) = \text{lead}(q_2(\mathbf{z}), k-1) = n, \deg(p_2(\mathbf{z}); k-1) = 0.$$

In q_1 the linear terms containing z_{k-1} are

$$(32) \quad z_{k-1} - z_k, z_1 + z_{k-1} - z_k, z_{k-1} - z_{\pi_{k-1}}$$

The first term here cancels with the identical term in the Vandermonde in p_1 . The second term divides Q_π , according to the following proposition from [9] applied for $l = k-1$:

Proposition 6.6 ([9], Proposition 7.4). *Let $l \geq 1$, and let π be an admissible sequence of partitions of the form $\pi = (\pi_1, \dots, \pi_l, [l+1], \dots, [k])$, where $\pi_l \neq [l]$. Then for $m > l$, and every partition τ such that $l \in \tau$, $\text{sum}(\tau) \leq m$, and $|\tau| > 1$, we have*

$$(33) \quad (z_\tau - z_m) | Q_\pi.$$

Therefore, after cancellation, all linear factors from $q_1(\mathbf{z})$ which have nonzero coefficients in front of both z_{k-1} and z_k vanish, and for the new fraction $\frac{p'_1(\mathbf{z})}{q'_1(\mathbf{z})}$

$$\deg(q'_1(\mathbf{z}), k-1) = \text{lead}(q'_1(\mathbf{z}), k-1) = 1.$$

By (32) and (14) $\deg(Q_\pi, k-1) \leq 3$ and therefore after cancellation we have

$$\deg(p'_1(\mathbf{z}), k-1) \leq k-2+2 = k$$

Using (31) we get

$$\deg(p'_1(\mathbf{z})p_2(\mathbf{z}), k-1) = k \text{ and } \deg(q'_1(\mathbf{z})q_2(\mathbf{z}), k-1) = \text{lead}(q'_1(\mathbf{z})q_2(\mathbf{z}), k-1) = n+1,$$

so we can apply the second option in Proposition 6.5 with $l = k-1$ to deduce the vanishing of the residue with respect to $k-1$.

In general, assume that

$$\pi = (\pi_1, \pi_2, \dots, \pi_l, [l+1], \dots, [k]) \text{ where } \pi_l \neq [l],$$

and proceed to the study of the residue with respect to z_l . For the second fraction we have again

$$(34) \quad \deg(q_2(\mathbf{z}), l) = \text{lead}(q_2(\mathbf{z}), l) = n, \deg(p_2(\mathbf{z}); l) = 0.$$

The linear terms containing z_l in $q_1(\mathbf{z})$ are

$$(35) \quad z_l - z_k, z_l - z_{k-1}, \dots, z_l - z_{l+1}$$

$$(36) \quad z_\tau - z_s \text{ with } l \in \tau, \tau \neq l, l+1 \leq s \leq k, \text{sum}(\tau) \leq s$$

$$(37) \quad z_l - z_{\pi_l}$$

The weights in (35) cancel out with the identical terms of the Vandermonde in $p_1(\mathbf{z})$ and by Proposition 6.6 $Q_\pi(\mathbf{z})$ is divisible by the weights in (36). Hence all linear factors with nonzero coefficient in front of z_l and at least one of z_{l+1}, \dots, z_k vanish from $q_1(\mathbf{z})$. Let again $\frac{p'_1(\mathbf{z})}{q'_1(\mathbf{z})}$ denote

the new fraction arising from $\frac{p_1(\mathbf{z})}{q_1(\mathbf{z})}$ after these cancellations. Then in $q'_1(\mathbf{z})$ only the term (37) contains z_l and

$$(38) \quad \deg(q'_1(\mathbf{z}), l) = \text{lead}(q'_1(\mathbf{z}), l) = 1.$$

In $p'_1(\mathbf{z})$ the linear terms which are left from the Vandermonde after cancellation and contain z_l are $z_{l-1} - z_l, \dots, z_1 - z_l$. The reduced $Q'_\pi(\mathbf{z})$ which we get after dividing by the terms in (36) is then a polynomial of the remaining weights, and the only remaining weights which contain z_l are

$$z_l - z_{\pi_l} \text{ and } z_l - z_k, z_l - z_{k-1}, \dots, z_l - z_{l+1}.$$

Then (14) tells us that $\deg(Q_\pi(\mathbf{z}), l) \leq k - l + 1$. Therefore

$$(39) \quad \deg(p'_1(\mathbf{z}); l) \leq (l - 1) + (k - l + 1) = k.$$

Putting (38) and (39) together we get

$$\deg(p'_1(\mathbf{z})p_2(\mathbf{z}), l) = k \text{ and } \deg(q'_1(\mathbf{z})q_2(\mathbf{z}), l) = \text{lead}(q'_1(\mathbf{z})q_2(\mathbf{z}), k - 1) = n + 1.$$

Since $k \leq n - 1$, by applying the second option of Proposition 6.5 we arrive at the vanishing of the residue, forcing π_l to be $[l]$.

7. INCREASING THE NUMBER OF POINTS AND THE PROOF OF THEOREM 1.2

The Residue Vanishing Theorem provides a closed iterated residue formula for tautological integrals on $\widetilde{CX}_p^{[k+1]}$ in the case when $k + 1 \leq n$, that is, the number of points does not exceed the dimension of X . In this section we show how one can drop this very restrictive condition.

Recall that the test curve model in §3.1 establishes a $\text{GL}(n)$ -equivariant isomorphism of quasi-projective varieties

$$J_k^{\text{reg}}(1, n) / J_k^{\text{reg}}(1, 1) \simeq CX_p^{[k+1]} \subset \text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$$

between the moduli of k -jets of regular germs and the curvilinear locus of the punctual Hilbert scheme sitting in the Grassmannian of k -dimensional subspaces in $\text{Sym}^{\leq k} \mathbb{C}^n$. For punctual Hilbert schemes we can assume without loss of generality that $X = \mathbb{C}^n$ and $p = 0$ and we use the notation

$$\text{CHilb}_0^{k+1}(\mathbb{C}^n) \subset \text{Hilb}_0^{k+1}(\mathbb{C}^n)$$

the curvilinear locus sitting in the punctual Hilbert scheme at the origin and $\overline{\text{CHilb}}_0^{k+1}(\mathbb{C}^n)$ for its closure, the curvilinear component.

Assume that $k + 1 > \dim(X) = n$. Fix a basis $\{e_1, \dots, e_k\}$ of \mathbb{C}^k and let

$$\mathbb{C}_{[n]} = \text{Span}(e_1, \dots, e_n) \hookrightarrow \mathbb{C}^{k+1} \text{ and } \mathbb{C}_{[k+1-n]} = \text{Span}(e_{n+1}, \dots, e_{k+1}) \hookrightarrow \mathbb{C}^{k+1}$$

denote the subspaces spanned by the first n and last $k + 1 - n$ basis vectors respectively. These are T_{k+1} -equivariant embeddings under the diagonal action of the maximal torus $T_{k+1} \subset \text{GL}(k + 1)$ and they induce the T_{k+1} -equivariant embedding

$$(40) \quad J_k^{\text{reg}}(1, n) \hookrightarrow J_k^{\text{reg}}(1, k + 1) = J_k^{\text{reg}}(1, n) \oplus \text{Hom}(\mathbb{C}^k, \mathbb{C}_{[k+1-n]}).$$

defined via $f \mapsto (f, 0)$. Here, by placing $f^{(i)}/i! \in \mathbb{C}^n$ into the i th column we identify $J_k^{\text{reg}}(1, n)$ with $\text{Hom}^{\text{reg}}(\mathbb{C}^k, \mathbb{C}^n)$, the set of k -by- n matrices with nonzero first column and the decomposition (40) reads as

$$(41) \quad J_k^{\text{reg}}(1, k+1) = \text{Hom}^{\text{reg}}(\mathbb{C}^k, \mathbb{C}^{k+1}) = \text{Hom}^{\text{reg}}(\mathbb{C}^k, \mathbb{C}_{[n]}) \oplus \text{Hom}(\mathbb{C}^k, \mathbb{C}_{[k+1-n]}) = \\ = J_k^{\text{reg}}(1, n) \oplus \text{Hom}(\mathbb{C}^k, \mathbb{C}_{[k+1-n]}).$$

Moreover, $J_k^{\text{reg}}(1, n)$ is invariant under the reparametrisation group $J_k^{\text{reg}}(1, 1)$ acting on $J_k^{\text{reg}}(1, k+1)$ and this action commutes with the T_{k+1} action resulting a T_{k+1} -equivariant embedding

$$\text{CHilb}_0^{k+1}(\mathbb{C}^n) \simeq J_k^{\text{reg}}(1, n)/J_k^{\text{reg}}(1, 1) \subset J_k^{\text{reg}}(1, k+1)/J_k^{\text{reg}}(1, 1) = \text{CHilb}_0^{k+1}(\mathbb{C}^{k+1}).$$

This embedding extends to the closures and commutes with the embeddings into the Grassmannians resulting the diagram

$$\begin{array}{ccc} \overline{\text{CHilb}}_0^{k+1}(\mathbb{C}^n) & \xrightarrow{T_{k+1}\text{-equiv}} & \overline{\text{CHilb}}_0^{k+1}(\mathbb{C}^{k+1}) \\ \downarrow \text{GL}(n)\text{-equiv} & & \downarrow \text{GL}(k+1)\text{-equiv} \\ \text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^n) & \xrightarrow{T_{k+1}\text{-equiv}} & \text{Grass}_k(\text{Sym}^{\leq k} \mathbb{C}^{k+1}) \end{array}$$

where the horizontal maps are T_{k+1} -equivariant and the vertical embeddings are $\text{GL}(n)$ resp. $\text{GL}(k+1)$ -equivariant.

The decomposition (41) induces a T_{k+1} -equivariant isomorphism of quasi-projective varieties

$$\Psi^{k+1 \rightarrow n} : J_k^{\text{reg}}(1, n)/J_k^{\text{reg}}(1, 1) \times \text{Hom}(\mathbb{C}^k, \mathbb{C}_{[k+1-n]}) \xrightarrow{\simeq} J_k^{\text{reg}}(1, k+1)/J_k^{\text{reg}}(1, 1) \\ (f_1 \cdot J_k^{\text{reg}}(1, 1), f_2) \mapsto (f_1 \oplus f_2) \cdot J_k^{\text{reg}}(1, 1)$$

whose inverse on the open chart

$$J_k^i(1, k+1)/J_k^{\text{reg}}(1, 1) = \{(f', \dots, f^{[k]}) \cdot J_k^{\text{reg}}(1, 1) : f'_i \neq 0\}$$

where the i th coordinate of f' does not vanish can be given using a canonical slice of the action given by the following

Lemma 7.1. *Let $n \geq 2$ and $f = (f', \dots, f^{[k]}) \in J_k^i(1, k+1)$. The $J_k^{\text{reg}}(1, 1)$ -orbit of f contains a unique point $\tilde{f} = (\tilde{f}', \dots, \tilde{f}^{[k]})$ such that $\tilde{f}'_i = 1$ and $\tilde{f}_i^{[j]} = 0$ for $j \geq 2$.*

Proof. $J_k^i(1, k+1)$ consists of $k+1$ -by- k matrices $(f', \dots, f^{(k)})$ whose $(i, 1)$ entry f'_i is nonzero $f'_i \neq 0$. The action of $J_k^{\text{reg}}(1, 1)$ is right multiplication with the matrix given in (4). This action multiplies the first column f' with α_1 whereas the image of $f^{(j)}$ for $2 \leq j \leq k$ is

$$\alpha_j f' + \sum_{\substack{\tau \in \mathcal{P}(j) \\ |\tau|=s}} \alpha_\tau \cdot f^{(s)}.$$

We choose the free parameter α_j inductively as $\alpha_1 = 1/f'_i$ and $\alpha_j = -\frac{1}{f'_i} \sum_{\substack{\tau \in \mathcal{P}(j) \\ |\tau|=s}} \alpha_\tau \cdot f_i^{(s)}$ for $2 \leq j \leq k$ to get the desired form of the matrix. \square

The T_{k+1} -equivariant inverse of $\Psi^{k+1 \rightarrow n}$ on the open chart $J_k^i(1, k+1)/J_k^{\text{reg}}(1, 1)$ is then given as

$$\begin{aligned} \Psi_i^{n \rightarrow k+1} : J_k^i(1, k+1)/J_k^{\text{reg}}(1, 1) &\rightarrow J_k^{\text{reg}}(1, n)/J_k^{\text{reg}}(1, 1) \oplus \text{Hom}(\mathbb{C}^k, \mathbb{C}_{[k+1-n]}) \\ (f_1 \oplus f_2) \cdot J_k(1, 1) &\mapsto (f_1 \cdot J_k(1, 1), \tilde{f}_2). \end{aligned}$$

This T_{k+1} -equivariant isomorphism gives us

Proposition 7.2. *For any T_{k+1} -equivariantly closed compactly supported form μ on $J_k^{\text{reg}}(1, k+1)/J_k^{\text{reg}}(1, 1)$ we have*

$$\int_{J_k^{\text{reg}}(1, n)/J_k^{\text{reg}}(1, 1)} \mu = \int_{J_k^{\text{reg}}(1, k+1)/J_k^{\text{reg}}(1, 1)} \mu \cdot \text{Euler}^{T_k}(\text{Hom}(\mathbb{C}^k, \mathbb{C}_{[k+1-n]}))$$

where $\text{Euler}^{T_k}(\text{Hom}(\mathbb{C}^k, \mathbb{C}_{[k+1-n]}))$ is the T_{k+1} -equivariant Euler class of $\text{Hom}(\mathbb{C}^k, \mathbb{C}_{[k+1-n]})$.

Proof. We use the topological definition of equivariant duals, see Remark 5.5 and [21, 27, 15] for details. We use the shorthand notations $\mathcal{J}_{k+1} = J_k^{\text{reg}}(1, k+1)/J_k^{\text{reg}}(1, 1)$ and $\mathcal{J}_n = J_k^{\text{reg}}(1, n)/J_k^{\text{reg}}(1, 1)$. The key observation is that $ET_{k+1} \times_{T_k} \mathcal{J}_n$ forms the zero section of the T_{k+1} -equivariant bundle $ET_{k+1} \times_{T_{k+1}} \mathcal{J}_{k+1}$ with fibres isomorphic to $\text{Hom}(\mathbb{C}^k, \mathbb{C}_{[k+1-n]})$. The (ordinary) Poincaré dual of the zero section is given by the top Chern class of the bundle, that is the T_{k+1} -equivariant Euler class of the fibre and therefore

$$\int_{ET_{k+1} \times_{T_{k+1}} \mathcal{J}_n} \mu = \int_{ET_{k+1} \times_{T_{k+1}} \mathcal{J}_{k+1}} \mu \cdot c^{\text{top}}$$

for any compactly supported equivariantly closed $\mu \in H^*(ET_k \times_{T_k} \mathcal{J}_k)$. \square

As a corollary we get the following

Corollary 7.3 (Extended Residue Vanishing Theorem). *Formula (25) remains valid for any $2 \leq n < k+1$.*

Proof. The weights of the T_{k+1} action on $\text{Hom}(\mathbb{C}^k, \mathbb{C}_{[k+1-n]})$ in Proposition 7.2 are the weights of the T_{k+1} action on $f_j^{[i]}$ for $1 \leq i \leq k$ and $n+1 \leq j \leq k+1$. The embedding $\phi^{\text{Flag}} : J_k^{\text{reg}}(1, k+1)/J_k^{\text{reg}}(1, 1) \hookrightarrow \text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^{k+1})$ is T_{k+1} -equivariant, and over the flag \mathbf{f}_σ the weight of $f_j^{[i]}$ is $\lambda_{\sigma(i)} - \lambda_{\sigma(j)}$. In the iterated residue formula of Proposition 5.10 we write $\lambda_i - z_j$ for this weight and therefore the T_{k+1} -equivariant Euler class transforms into

$$\text{Euler}_{\mathbf{z}}^{T_{k+1}}(\text{Hom}(\mathbb{C}^k, \mathbb{C}_{[k+1-n]}^{\mathbf{z}})) = \prod_{i=1}^k \prod_{j=n+1}^k (\lambda_j - z_i)$$

over the flag \mathbf{f}_σ corresponding to an iterated pole $\mathbf{z} = (z_1, \dots, z_k)$. If $\alpha = \alpha(\theta_1, \dots, \theta_r, \eta_1, \dots, \eta_k)$ is a bi-symmetric polynomial in the Chern roots θ_i of the pull-back of F over $\widehat{CX}_p^{[k+1]} = \text{CHilb}_0^{k+1}(\mathbb{C}^n) \subset \text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^n)$ and the Chern roots η_j of the tautological rank k bundle \mathcal{E} , then α is the restriction of a closed form on $\text{Flag}_k(\text{Sym}^{\leq k} \mathbb{C}^{k+1})$ and in particular it is a

restriction of a form on $\text{CHilb}_0^{k+1}(\mathbb{C}^{k+1})$. Therefore Remark 6.3, Proposition 7.2 and Theorem 6.1 tell us that

$$\begin{aligned} \int_{\overline{CX}_p^{[k+1]}} \alpha &= \text{Res}_{\mathbf{z}=\infty} \frac{Q_k(\mathbf{z}) \prod_{m < l} (z_m - z_l) \alpha(\theta, \mathbf{z}) d\mathbf{z}}{\prod_{m+r \leq l \leq k} (z_m + z_r - z_l) \prod_{l=1}^k \prod_{i=1}^k (\lambda_i - z_l)} \cdot \prod_{i=1}^k \prod_{j=n+1}^k (\lambda_j - z_i) = \\ &= \text{Res}_{\mathbf{z}=\infty} \frac{Q_k(\mathbf{z}) \prod_{m < l} (z_m - z_l) \alpha(\theta, \mathbf{z})}{\prod_{m+r \leq l \leq k} (z_m + z_r - z_l) \prod_{l=1}^k \prod_{i=1}^k (\lambda_i - z_l)} d\mathbf{z}. \end{aligned}$$

□

7.1. Proof of Theorem 1.2 and final remarks. The weights $\lambda_1, \dots, \lambda_n$ are the Chern roots of T_p^*X and therefore $-\lambda_1, \dots, -\lambda_n$ are the weights on T_pX . Theorem 1.2 follows from the Residue Vanishing Theorem by substituting

$$\frac{1}{\prod_{i=1}^n (\lambda_i - z_j)} = \frac{(-1)^n}{z_j^n c(1/z_j)} = (-1)^n \frac{s_X(1/z_j)}{z_j^n}$$

If we give the z_i 's and θ_j 's degree 1 then the total degree of the rational expression

$$\frac{(-1)^{nk} \prod_{i < j} (z_i - z_j) Q_k(z) M(c_i(z_i + \theta_j, \theta_j))}{\prod_{i+j \leq k} (z_i + z_j - z_l)(z_1 \dots z_k)^n}$$

in the formula is $n - k$.

The Chern class $c_i(z_i + \theta_j, \theta_j)$ is the coefficient of t^i in

$$c(F^{[k+1]})(t) = \prod_{j=1}^r (1 + \theta_j t) \prod_{i=1}^k \prod_{j=1}^r (1 + z_i t + \theta_j t),$$

that is, the i th Chern class of the bundle with formal Chern roots $\theta_j, z_i + \theta_j$. For example

$$c_1(z_i + \theta_j, \theta_j) = (k+1) \sum_{j=1}^r \theta_j + r \sum_{i=1}^k z_i,$$

and in general $c_i(z_i + \theta_j, \theta_j)$ is a degree i polynomial of the form

$$c_i(z_i + \theta_j, \theta_j) = A_i c_i(\mathbf{z}) + A_{i-1} c_{i-1}(\mathbf{z}) + \dots + A_0$$

where $c_j(\mathbf{z})$ is the j th elementary symmetric polynomial in z_1, \dots, z_k and A_j is a degree $n - j$ symmetric polynomial in $\theta_1, \dots, \theta_r$.

REFERENCES

- [1] Arnold, V. I., Goryunov, V. V., Lyashko, O. V., Vasilliev, V. A.: Singularity theory I. Dynamical systems VI, Encyclopaedia Math. Sci., Springer-Verlag, Berlin (1998)
- [2] Atiyah M., Bott, R.: The moment map and equivariant cohomology. Topology 23(1), 1-28 (1984)
- [3] Bérczi, G.: Moduli of map germs, Thom polynomials and the Green-Griffiths conjecture. Contributions to Algebraic Geometry, edited by P. Pragacz, EMS 141-168 (2012)
- [4] Bérczi, G.: Multidegrees of singularities and non-reductive quotients. PhD Thesis, Eötvös University Budapest (2008)

- [5] Bérczi, G., Fehér, L.M., Rimányi, R.: Expressions for resultants coming from the global theory of singularities. *Topics in Algebraic and Noncommutative Geometry*, Contemporary Mathematics 324, 63-69 (2003)
- [6] Bérczi, G.: Thom polynomials of Morin singularities and the Green-Griffiths-Lang conjecture, arXiv:1011.4710
- [7] Bérczi, G. and Kirwan, F., Grosshans theory for graded unipotent groups, in preparation
- [8] Bérczi, G., Doran, B., Hawes, T., Kirwan, F., Geometric invariant theory for graded unipotent groups and applications, in preparation
- [9] Bérczi, G., Szenes, A.: Thom polynomials of Morin singularities. *Annals of Mathematics* 175, 567-629 (2012)
- [10] Berline, H., Getzler, E., Vergne, M.: Heat kernels and Dirac operators. Springer-Verlag Berlin (2004)
- [11] Berline, N., Vergne, M.: Zeros dun champ de vecteurs et classes caracteristiques equivariantes. *Duke Math. J.* 50(2), 539-549 (1973)
- [12] Bott, R. and Tu, L. W. , Differential forms in algebraic topology, Graduate Texts in Mathematics, Springer-Verlag, 1982.
- [13] Demailly, J.-P.: Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials. *Proc. Sympos. Pure Math.* 62, 285-360 (1982)
- [14] Doran, B., Kirwan, F.: Towards non-reductive geometric invariant theory. *Pure and Appl. Math. Q.* 3 , 61-105 (2007)
- [15] Edidin, D. and Graham, W.: Characteristic classes in the Chow ring. *J. Algebraic Geom.*, 6(3):431-443, 1997.
- [16] Ellingsrud, G., Göttsche, L. and Lehn, M., On the cobordism class of the Hilbert scheme of a surface. *J. Algebraic Geom.*, 10(1):81-100, 2001.
- [17] Eisenbud, D.: Commutative algebra, with a view toward algebraic geometry. Graduate Texts in Mathematics 150, Springer-Verlag (1995)
- [18] Fehér, L. M., Rimányi, R.: Thom polynomial computing strategies. A survey. *Adv. Studies in Pure Math.* 43, Singularity Theory and Its Applications, Math. Soc. Japan, 45-53 (2006)
- [19] Fehér, L. M., Rimányi, R.: Thom series of contact singularities. *Annals of Mathematics* 176, 1381-1426 (2012)
- [20] Fulton, W.: Intersection Theory. Springer-Verlag, New York (1984)
- [21] Fulton, W.: Equivariant cohomology in algebraic geometry. <http://www.math.lsa.umich.edu/~danderson/eilenberg>. Eilenberg lectures, Columbia University, Spring 2007.
- [22] Gaffney, T.: The Thom polynomial of P^{111} . *Proc. Symp. Pure Math.* 40, 399-408 (1983)
- [23]
- [24] Green, M., Griffiths, P.: Two applications of algebraic geometry to entire holomorphic mappings. *The Chern Symposium 1979. Proc. Intern. Symp.* 41-74, Springer, New York (1980)
- [25] Haefliger, A., Kosinski, A.: Un théorème de Thom sur les singularités des applications différentiables. *Séminaire Henri Cartan 9 Exposé 8* (1956-57)
- [26] Joseph, A.: On the variety of a highest weight module. *J. of Algebra* 88, 238-278 (1984)
- [27] Kazarian, M.: Characteristic classes of singularity theory. In *The Arnold-Gelfand mathematical seminars*, pages 325-340. Birkhäuser Boston, 1997.
- [28] Kazarian, M.: Thom polynomials for Lagrange, Legendre, and critical point function singularities. *Proc. LMS* 86 707-734 (2003)
- [29] Lehn, M., Chern classes of tautological sheaves on Hilbert schemes of points on surfaces, *Invent. Math.* 136 (1999), 157-207.
- [30] Marian, A., Oprea, D., Pandharipande, R., Segre classes and Hilbert schemes of points, arXiv:1507.00688.
- [31] Miller, E., Sturmfels, B.: Combinatorial Commutative Algebra. Springer Verlag, Berlin (2004)
- [32] Mumford, D., Fogarty, J., Kirwan, F.: Geometric Invariant Theory. Springer Verlag, Berlin (1994)
- [33] Nakajima, H., Heisenberg algebra and Hilbert schemes of points on projective surfaces, *Ann. Math.* 145 (1997), 379-388.

- [34] Rennemo, J.: Universal Polynomials for Tautological Integrals on Hilbert Schemes, arXiv:1205.1851 (2012).
- [35] Rimányi, R.: Thom polynomials, symmetries and incidences of singularities. *Invent. Math.* 143(3), 499-521 (2001)
- [36] Rossmann, W.: Equivariant multiplicities on complex varieties. *Orbites unipotentes et representations, III. Asterisque No. 173-174*, 313-330 (1989)
- [37] Szenes, A.: Iterated residues and multiple Bernoulli polynomials. *Int. Math. Res. Not.* 18, 937-956 (1998)
- [38] Thom, R.: Les singularités des applications différentiables. *Ann. Inst. Fourier* 6, 43-87 (1955-56)
- [39] Vergne, M.: Polynomes de Joseph et représentation de Springer. *Annales scientifiques de l'École Normale Supérieure* 23.4, 543-562 (1990)

MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD, ANDREW WILES BUILDING, OX2 6GG OXFORD, UK
E-mail address: berczi@maths.ox.ac.uk